



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

## Global well-posedness of the derivative nonlinear Schrödinger equation with periodic boundary condition in $H^{1/2}$

### Citation for published version:

Mosincat, R 2017, 'Global well-posedness of the derivative nonlinear Schrödinger equation with periodic boundary condition in  $H^{1/2}$ ', *Journal of Differential Equations*, vol. 263, no. 8, pp. 4658 - 4722.  
<https://doi.org/10.1016/j.jde.2017.05.026>

### Digital Object Identifier (DOI):

[10.1016/j.jde.2017.05.026](https://doi.org/10.1016/j.jde.2017.05.026)

### Link:

[Link to publication record in Edinburgh Research Explorer](#)

### Document Version:

Peer reviewed version

### Published In:

Journal of Differential Equations

### General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

### Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact [openaccess@ed.ac.uk](mailto:openaccess@ed.ac.uk) providing details, and we will remove access to the work immediately and investigate your claim.



# GLOBAL WELL-POSEDNESS OF THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION WITH PERIODIC BOUNDARY CONDITION IN $H^{\frac{1}{2}}$

RAZVAN MOSINCAT

*School of Mathematics, The University of Edinburgh  
and The Maxwell Institute for Mathematical Sciences  
James Clerk Maxwell Building, The King's Buildings  
Peter Guthrie Tait Road, EH9 3FD, Edinburgh, United Kingdom*

ABSTRACT. We establish the global well-posedness of the derivative nonlinear Schrödinger equation with periodic boundary condition in the Sobolev space  $H^{\frac{1}{2}}$ , provided that the mass of initial data is less than  $4\pi$ . This result matches the one by Miao, Wu, and Xu and its recent mass threshold improvement by Guo and Wu in the non-periodic setting. Below  $H^{\frac{1}{2}}$ , we show that the uniform continuity of the solution map on bounded subsets of  $H^s$  does not hold, for any gauge equivalent equation.

## CONTENTS

1. Introduction	2
2. Notations and basic estimates	8
2.1. A bilinear $L^4$ -Strichartz estimate	12
2.2. Gagliardo-Nirenberg inequalities in the periodic setting	14
2.3. The gauge transformation on $\mathbb{T}_\lambda$	15
2.4. The $I$ -operator	17
2.5. Multilinear forms	18
3. Coercivity properties	20
4. Local well-posedness for the $I$ -system	24
5. Modified energy functionals via the $I$ -operator and correction terms	27
5.1. Pointwise bounds on the multipliers	30
5.2. Necessity of the third iteration of the $I$ -method	31
5.3. The third generation modified energy	34
5.4. A non-resonant set for $\alpha_6$	35
5.5. Pointwise bounds on the multipliers (continued)	36
6. Almost conservation estimates for the third generation modified energy	37
7. Control of the almost conserved energy and of the almost conserved momentum	46
8. Proof of Proposition 2.14 via the $I$ -method	48

---

*E-mail address:* r.o.mosincat@sms.ed.ac.uk.

*Date:* May 23, 2017.

*2010 Mathematics Subject Classification.* 35Q55.

*Key words and phrases.* derivative nonlinear Schrödinger equation; global well-posedness.

Appendix A. Mild ill-posedness below $H^{\frac{1}{2}}(\mathbb{T})$	50
References	52

## 1. INTRODUCTION

We consider the initial value problem for the derivative nonlinear Schrödinger equation (DNLS)<sup>1</sup> with periodic boundary condition

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u), & (t, x) \in \mathbb{R} \times \mathbb{T} \\ u(0, x) = u_0(x), & u_0 \in H^s(\mathbb{T}) \end{cases}, \quad (1.1)$$

where  $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z} \simeq [0, 2\pi)$ . This equation was derived in the plasma physics literature in the 1970s [37, 32, 33] and it is a particular case of a perturbed complex Ginzburg-Landau equation [46]. Kaup and Newell [19] showed that (1.1) is completely integrable, in the sense that it is the compatibility condition for a certain pair of linear differential equations. In particular, it possesses an infinite family of conservation laws, as well as a two-parameter family of solitons. In this work, we only employ the conservation of the following integrals of motion, referred to as the *mass*, *momentum*, and *energy* (of a solution  $u$ ), respectively:

$$M[u] := \int |u|^2 dx, \quad (1.2)$$

$$P[u] := \int \operatorname{Im}(u \partial_x \bar{u}) + \frac{1}{2} |u|^4 dx, \quad (1.3)$$

$$E[u] := \int |\partial_x u|^2 + \frac{3}{2} |u|^2 \operatorname{Im}(u \partial_x \bar{u}) + \frac{1}{2} |u|^6 dx. \quad (1.4)$$

We note that these quantities are at the levels  $L^2$ ,  $H^{\frac{1}{2}}$ , and  $H^1$ , respectively; the Hamiltonian for (1.1) is  $P[u]$ . Given a sufficiently regular solution  $u$ , one can check by direct computation the conservation of the above functionals (see for example [17, Appendix B]). In addition, one can similarly verify that the *mean*  $\int u(t, x) dx$  is also conserved.

The scaling symmetry of this equation in the Euclidean setting is given by the invariance of solutions under the following transformation:

$$u(t, x) \mapsto \frac{1}{\lambda^{\frac{1}{2}}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) =: u^\lambda(t, x). \quad (1.5)$$

In particular,  $\|u^\lambda(t)\|_{L_x^2} = \|u(t/\lambda^2)\|_{L_x^2}$ , and thus  $s_c = 0$  is the scaling critical Sobolev index (reasonable well-posedness theory is to be expected for  $s > s_c$ , and possibly for  $s = s_c$ ). In the periodic setting, the above transformation changes the underlying domain  $\mathbb{T}$  to  $\mathbb{T}_\lambda := \mathbb{R}/2\pi\lambda\mathbb{Z}$  (hence, the scaling transformation above is no longer a symmetry of the equation on  $\mathbb{T}$ ). Nevertheless, we employ (1.5) in this article and for the most part, we work on the dilated torus  $\mathbb{T}_\lambda$ , with  $\lambda \geq 1$ .

The aim of this paper is to study the long-time dynamics of low-regularity solutions of (1.1). The main result of this work reads as follows:

**Theorem 1.1.** *The derivative nonlinear Schrödinger equation with periodic boundary condition (1.1) is globally well-posed in  $H^{\frac{1}{2}}$  for initial data  $u_0$  with  $M[u_0] < 4\pi$ .*

---

1. A parameter in front of the nonlinearity is irrelevant in this study since on the Fourier side the nonlinearity is unsigned; the equation does not have a definite focusing or defocusing character.

Before delving into the prerequisites and proof of this theorem, let us promptly review the well-posedness results in the Euclidean setting. For a certain class of Schwartz initial data, by using the inverse scattering method, Lee [25, 26] obtained the local and global<sup>2</sup> solvability, respectively. Tsutsumi and Fukuda [44] established the local existence and uniqueness of  $H^s$ -solutions, with  $s > \frac{3}{2}$ , by the method of parabolic regularization. Furthermore, in [45], they obtained the global existence of solutions for  $u_0 \in H^2$  with sufficiently small  $H^1$ -norm.

In the energy space, Hayashi [14] proved the global existence for sufficiently small initial data. In his work, of particular importance is the gauge transformation defined by

$$\mathcal{G}_\beta : L^2 \rightarrow L^2, \quad \mathcal{G}_\beta f(x) := e^{-i\beta \mathcal{J}(f)(x)} f(x), \quad (1.6)$$

with  $\mathcal{J}(f)(x) := \int_{-\infty}^x |f(y)|^2 dy$ ,  $x \in \mathbb{R}$ . Through the transformation  $v = \mathcal{G}_{\frac{1}{2}}(u)$ , DNLS reduces to

$$i\partial_t v + \partial_x^2 v = i|v|^2 \partial_x v \quad (1.7)$$

and the theory of [45] for smooth initial data with sufficiently small  $H^1$ -norm can be applied. This result was improved in the papers by Hayashi and Ozawa [15, 16] by reducing DNLS to a system of two semi-linear Schrödinger equations (with no derivatives in the nonlinearities), where it was obtained<sup>3</sup> the global existence of  $H^1$ -solutions under the assumption

$$M[u_0] < 2\pi. \quad (1.8)$$

This mass threshold follows from noticing that under the transformation  $v = \mathcal{G}_{\frac{3}{4}}(u)$ , the energy functional becomes<sup>4</sup>

$$E[u] = E[\mathcal{G}_{-\frac{3}{4}}(v)] = \|\partial_x v\|_{L^2(\mathbb{R})}^2 - \frac{1}{16} \|v\|_{L^6(\mathbb{R})}^6, \quad (1.9)$$

while the norms of interest remain essentially unchanged, i.e.

$$\|\partial_x v\|_{L^2(\mathbb{R})} \sim_{M[u]} \|\partial_x u\|_{L^2(\mathbb{R})}, \quad (1.10)$$

$$\|v\|_{L^p(\mathbb{R})} = \|u\|_{L^p(\mathbb{R})}. \quad (1.11)$$

Via the sharp Gagliardo-Nirenberg inequality due to Weinstein [47], i.e.

$$\|v\|_{L^6(\mathbb{R})}^6 \leq \frac{4}{\pi^2} \|\partial_x v\|_{L^2(\mathbb{R})}^2 \|v\|_{L^2(\mathbb{R})}^4, \quad (1.12)$$

one obtains that the energy  $E[u]$  controls the  $\dot{H}^1$ -norm of a solution  $u$  of DNLS, provided that (1.8) holds. Finally, we mention that in [14, 15], uniqueness of  $H^1(\mathbb{R})$ -solutions was also obtained, but conditional to the auxiliary spaces  $L_t^{12}(\mathbb{R}; H_x^{1,3}(\mathbb{R}))$ ,  $L_t^4(\mathbb{R}; W_x^{1,\infty}(\mathbb{R}))$ , respectively. The unconditional well-posedness in  $H^1(\mathbb{R})$  was settled by Win [48].

In low-regularity spaces, Takaoka [38] used the Fourier restriction norm spaces introduced by Bourgain [3] and proved

$$\|v^2 \partial_x \bar{v}\|_{X^{s,b-1}(\mathbb{R} \times \mathbb{R})} \lesssim \|v\|_{X^{s,b}(\mathbb{R} \times \mathbb{R})}^3, \quad (1.13)$$

for  $\frac{1}{2} \leq s < 1$  and  $\frac{1}{2} < b \leq \frac{5}{8}$ , in a fashion similar to the estimate for the KdV equation [20]. We remind the reader that the Fourier restriction norms  $\|\cdot\|_{X^{s,b}(\mathbb{R} \times \mathbb{R})}$  are adapted to

2. We refer to the recent article of Pelinovsky and Shimabukuro [36, p. 5] for a possible issue regarding the result of [26].

3. Hayashi and Ozawa [15, 16] also showed that the solution map  $u_0 \mapsto u(t)$  preserves Sobolev regularity and spatial decay, for any  $t \in \mathbb{R}$ .

4. If  $u$  solves DNLS, then  $v = \mathcal{G}_{\frac{3}{4}}(u)$  solves  $i\partial_t v + \partial_x^2 v = \frac{i}{2}|v|^2 \partial_x v - \frac{i}{2}v^2 \partial_x \bar{v} - \frac{3}{16}|v|^4 v$ .

the linear part of the equation at hand (defined by (2.5) on  $\mathbb{T}$ , their definition on  $\mathbb{R}$  being analogous). On the other hand, Takaoka in [38] noted that for estimates of the form

$$\| |v|^2 \partial_x v \|_{X^{s,b-1}(\mathbb{R} \times \mathbb{R})} \lesssim \|v\|_{X^{s,b}(\mathbb{R} \times \mathbb{R})}^3, \quad (1.14)$$

“the Fourier restriction norm method seems inapplicable.” However, the transformation<sup>5</sup>  $v = \mathcal{G}_1(u)$  removes the nonlinearity  $|u|^2 \partial_x u$  from (1.1), i.e.  $v$  solves

$$i \partial_t v + \partial_x^2 v = -i v^2 \partial_x \bar{v} - \frac{1}{2} |v|^4 v. \quad (1.15)$$

Therefore,<sup>6</sup> Takaoka established the local well-posedness of (1.1) down to  $H^{\frac{1}{2}}(\mathbb{R})$ . Since the mappings  $u_0 \mapsto \mathcal{G}_1(u_0) = v_0$  and  $v(t) \mapsto \mathcal{G}_1^{-1}(v(t)) = u(t)$ , viewed as applications from  $H^{\frac{1}{2}}(\mathbb{R})$  to itself, are locally Lipschitz continuous (uniformly in  $t$ ), so is the dependence of  $u$  on the initial data  $u_0$ . Takaoka also showed that the above estimate (1.13) does not hold if  $s < \frac{1}{2}$ , for any  $b \in \mathbb{R}$ . Moreover, for  $0 \leq s < \frac{1}{2}$ , the solution map  $u_0 \in H^s(\mathbb{R}) \mapsto u(t) \in H^s(\mathbb{R})$  fails to be  $C^3$ , for any  $t \neq 0$ . Another mild ill-posedness result for DNLS in  $H^s(\mathbb{R})$  ( $0 \leq s < \frac{1}{2}$ ) was given by Biagioni and Linares in [2]. They used the solitary waves of DNLS [19, 46] and showed that the local uniform continuity of the same solution map does not hold. Hence, the fixed point argument for the gauge equivalent equation (1.15) is no longer the tool to construct  $H^s(\mathbb{R})$ -solutions for DNLS in the range  $0 \leq s < \frac{1}{2}$ .

Let us now elaborate on the assumption (1.8). The energy functional in (1.9) is also shared by the focusing quintic NLS (with some appropriate constant in front of the nonlinearity), for which the condition (1.8) is sharp, in the sense that finite-time blowup solutions with  $M[u] \geq 2\pi$  exist (see [47] and references therein). Thus, the question of whether the same is true for DNLS appears naturally (see also [15]). However, in a recent series of articles, Wu [50, 51], and Guo and Wu [13] obtained global existence for DNLS above the mass threshold  $2\pi$ . They showed how to incorporate the momentum  $P[u]$  in controlling the  $\dot{H}^1$ -norm of  $u$ . The key observation is the following: the change in energy incurred by modulating  $u$  resembles the first term of the momentum (see (1.3)), and then the second term of  $P[u]$  is handled by another sharp Gagliardo-Nirenberg inequality due to Agueh [1], that interpolates  $L^6(\mathbb{R})$  between  $\dot{H}^1(\mathbb{R})$  and  $L^4(\mathbb{R})$  (rather than  $L^2(\mathbb{R})$ ), i.e.

$$\|u\|_{L^6(\mathbb{R})} \leq C_{\text{GN}} \|\partial_x u\|_{L^2(\mathbb{R})}^{\frac{1}{9}} \|u\|_{L^4(\mathbb{R})}^{\frac{8}{9}}, \quad (1.16)$$

where  $C_{\text{GN}} = 3^{\frac{1}{6}}(2\pi)^{-\frac{1}{9}}$ . The upshot is the control of the  $\dot{H}^1$ -norm of a solution  $u$ , under

$$M[u_0] < 4\pi. \quad (1.17)$$

It is known that the mass thresholds  $2\pi$  and  $4\pi$  correspond to the masses of ground state solutions to some elliptic equations and extremising functions in the Gagliardo-Nirenberg inequalities (1.12) and (1.16), respectively. In a recent article studying the orbital stability of solitary waves of DNLS with mass  $4\pi$ , Kwon and Wu [22]<sup>7</sup> proposed a criterion for blowup solutions with this critical mass.

5. This reduction of DNLS to (1.15) was also employed by Lee [26], to which he attached a certain spectral problem.

6. The estimate for the quintic term of (1.15) is easier to prove than for the cubic-derivative term.

7. In [22] it is also shown the orbital stability of traveling waves in the “zero mass case”. Also, after the completion of this article, several papers studying the traveling waves for DNLS were published. See [23, 29, 30] (and references therein for previous works on this subject). In [29] there is an improvement, as compared to [51], of the subset of energy-space solutions that exist globally in time.

We now turn to the periodic setting. The adaptation of the gauge transformation is due to Herr [17] where he proved the local well-posedness of (1.1) in  $H^s(\mathbb{T})$  for  $s \geq \frac{1}{2}$ , by using the same transformation (1.6), but with  $\mathcal{J}(f)$  defined as the mean-zero antiderivative of  $|f|^2 - \frac{1}{2\pi}M[f]$ . Compared to the real line case, the gauge transformation is no longer enough to satisfactorily reduce DNLS to a manageable equation, and thus it needs to be augmented with a translation operator (see (2.41)). After these transformations, one works instead with a *periodic gauge equivalent derivative nonlinear Schrödinger equation*, that still resembles (1.15), of the form

$$i\partial_t v + \partial_x^2 v = -iv^2 \partial_x \bar{v} + \mathcal{Q}(v), \quad (1.18)$$

where  $\mathcal{Q}(v)$  gathers (pure-power) quintic and lower order terms (a precise formulation of the above equation is given by (2.43)). For the key estimate in [17], i.e.

$$\|v^2 \partial_x \bar{v}\|_{X^{s, -\frac{1}{2}}(\mathbb{R} \times \mathbb{T})} \lesssim \|v\|_{X^{s, \frac{1}{2}}(\mathbb{R} \times \mathbb{T})}^3 \quad (1.19)$$

for  $s \geq \frac{1}{2}$ , the local smoothing and maximal function estimates for the linear Schrödinger propagator are no longer available – one has to rely merely on the  $L^4$ -Strichartz estimate of Bourgain [3] and on Sobolev inequalities. Since the embedding of  $X^{s, \frac{1}{2}}(\mathbb{R} \times \mathbb{T})$  into  $C(\mathbb{R}; H^s(\mathbb{T}))$  fails, one works instead with a slightly stronger norm (see (2.6)-(2.7) below).

Regarding possible improvements to the estimate (1.19), a remarkable property of (1.18) was uncovered by Grünrock and Herr [12] while working in the scale of Fourier-Lebesgue spaces  $\mathcal{FL}^{s,r}(\mathbb{T})$  (defined by (2.2) below). By appropriately modifying the classical  $X^{s,b}$  norms, they established a local well-posedness result in  $\mathcal{FL}^{\frac{1}{2},r}(\mathbb{T})$ , in the range  $2 \leq r < 4$ . (We note that for  $s = \frac{1}{2}$ , these spaces scale like  $H^{\frac{1}{r}}(\mathbb{T})$ ). It turns out that the nonlinearity of (1.18) can be rearranged into  $\tilde{\mathcal{T}}(v) + \tilde{\mathcal{Q}}(v)$ , with

$$\tilde{\mathcal{T}}(v) := -i \left( v \partial_x \bar{v} - 2i \operatorname{Im} \int_{\mathbb{T}} v \partial_x \bar{v} dx \right) v, \quad (1.20)$$

owing to the periodic gauge transformation. This writing reveals frequency cancelations (see also (2.47)) that are essential for dealing with the cubic-derivative nonlinearity, and hence for the local well-posedness in  $\mathcal{FL}^{s,r}(\mathbb{T})$ . However, (1.20) does not help when trying to prove (1.19) for some  $s < \frac{1}{2}$ , as the failure of this estimate is of another nature than the lack of such frequency cancelations.

We also point out that, specific to the periodic case, the Lipschitz continuity of the solution map of DNLS on bounded subsets of  $H^s(\mathbb{T})$  is further restricted to subsets with prescribed  $L^2$ -norm due to the use of a translation operator when reversing the transformations leading to (1.18) back to (1.1) (see Lemma 2.13). In fact, the local uniform continuity of the solution map of the periodic DNLS fails without fixing the mass on bounded subsets of  $H^s(\mathbb{T})$ , at any regularity level (see [18, Theorem 3.1.1.(ii)]). Although counterexamples to the trilinear estimate (1.19) for  $s < \frac{1}{2}$  were given in [12], to the best of the author's knowledge, a direct argument towards the (mild) ill-posedness of (1.1) in  $H^s(\mathbb{T})$  can not be found in the literature. Note that for the gauge equivalent equation (1.18), one does not face the local uniform continuity bottleneck due to the translation operator and it was for this equation that the contraction mapping argument was applied in [17]. Hence, we provide here the following mild ill-posedness result. The mild sense refers to the fact that the result shows that the contraction mapping argument cannot be applied for the gauge equivalent equation (1.18). In fact, the same is true for any equation obtained from (1.1) through a gauge transformation of the form (1.6) (see Appendix A).

**Proposition 1.2.** *Suppose  $v_0 \in H^s(\mathbb{T}) \mapsto v =: S_T(v_0) \in C([-T, T]; H^s(\mathbb{T}))$  is the solution map for (1.18), for some  $0 \leq s < \frac{1}{2}$ ,  $T > 0$ . Then,  $S_T$  is not uniformly continuous on bounded subsets of  $H^s(\mathbb{T})$ .*

Turning our attention to global-in-time solutions of DNLS, we recall the reader that the global existence theory below  $H^1(\mathbb{R})$ , under the smallness of mass condition (1.8), was developed by Colliander, Keel, Staffilani, Takaoka, and Tao [7, 8]. First, in [7], they introduced the so called “ $I$ -method” which aims to control the growth of  $\mathcal{E}^1[u] := \mathcal{E}[Iu]$ , where  $I$  (defined in (2.50)) is a smoothing operator on large frequencies and it is the identity operator for functions supported on small frequencies (here,  $\mathcal{E}$  stands for the energy of (1.15) rather than of DNLS). Relying on the  $L^6$ -Strichartz estimate and on a bilinear  $L^4$ -Strichartz inequality (in addition to the usual Sobolev embeddings), they obtain the global existence in  $H^s(\mathbb{R})$ , for  $s > \frac{2}{3}$ . We also mention here that the precursor to this method, the high-low method of Bourgain, was also implemented to DNLS on the real line by Takaoka in [39], where global solutions for  $s > \frac{32}{33}$  were constructed. A stronger result was established in [8] for the second generation modified energy  $\mathcal{E}^2[u]$ , which is obtained from  $\mathcal{E}^1[u]$  via a small correction term, thus lowering the regularity to  $s > \frac{1}{2}$ . What is critical to the latter result is an improved trilinear estimate (1.13) in the presence of the  $I$ -operator that can reach  $b = \frac{3}{4} - \varepsilon$ , as well as delicate cancelations owing to the nonlinear structure  $-iv^2\partial_x\bar{v} - \frac{1}{2}|v|^4v$  of (1.15). Miao, Wu, and Xu [31] closed the gap between the local well-posedness range and the output of the  $I$ -method by iterating the scheme further via a third generation modified energy  $\mathcal{E}^3[u]$  by adding another correction term to  $\mathcal{E}^2[u]$ . This time, the correction term has a singular set more complicated than that of the correction term in [8], and thus it requires an intricate resonant decomposition. Finally, we mention here that recently, Guo and Wu [13] improved the mass threshold to (1.17) for global  $H^{\frac{1}{2}}(\mathbb{R})$ -solutions.

Regarding the global existence question for the periodic problem (1.1), Herr [17] answered in affirmative for  $H^1(\mathbb{T})$ -solutions, under the assumption that the mass is smaller than  $\frac{2}{3}$ . This followed routinely from iterating the local well-posedness result together with a coercivity property of  $E[\cdot]$  (obtained by using a non-optimal Gagliardo-Nirenberg inequality and without appealing to another gauge choice). In the present work, we also show that the mass threshold under which the energy functional associated to (1.1) has the coercivity property is the same as in the real line case (see Lemma 3.2 below). However, the smallness of mass condition for global  $H^1(\mathbb{T})$ -solutions was already improved to (1.17) by the author and Oh in [34]. Below the energy space, in  $H^s(\mathbb{T})$ , for  $s > \frac{1}{2}$ , and unquantified small mass initial data, the global existence of solutions to (1.1) was studied by Win in [49] by using a second generation modified energy for the  $I$ -method.

The main tools we use in obtaining Theorem 1.1 are the following: (i) the gauge transformation (and its Lipschitz continuity property) and the multi-linear estimates due to Herr [17], (ii) a third generation modified energy in the  $I$ -method scheme of Colliander, Keel, Staffilani, Takaoka, and Tao [7, 8], (iii) a resonant decomposition of one of the correction terms as implemented in the real line case by Miao, Wu, and Xu [31], (iv) a revised bilinear  $L^4$ -Strichartz estimate of De Silva, Pavlović, Staffilani, and Tzirakis [10], and (v) the sharp Gagliardo-Nirenberg inequalities (1.12), (1.16) adapted to the periodic setting.

Before outlining the contents of this article, we end the introduction with the following remarks.

**Remark 1.3.** Our method is an analytical argument, in particular we do not employ the integrable structure of (1.1) in an explicit manner. Therefore, the technique can be applied

(cf. [8, 17, 38, 41]) to equations of the form

$$i\partial_t u + \partial_x^2 u = i\alpha|u|^2\partial_x u + i\beta u^2\partial_x \bar{u} + \gamma|u|^4 u, \quad (1.21)$$

where  $\alpha, \beta, \gamma \in \mathbb{R}$ . The first term is eliminated by the gauge transformation  $v = \mathcal{G}_{\alpha/2}(u)$ .

**Remark 1.4.** Whether DNLS exhibits finite-time blowup solutions for initial data with large mass remains an important open question. A numerical study by Liu, Simpson and Sulem [28] indicates that there is finite time singularity for the  $L^2$ -supercritical nonlinearity  $i|u|^{2\sigma}\partial_x u$ ,  $\sigma > 1$ . We also refer to [5] for a further numerical investigation of the structure of the singular profile near blowup times. Here, we also point out that negative energy solutions to DNLS on a bounded interval or on the half-line, with Dirichlet boundary conditions, blow up in finite time (see [42], [50]).

**Remark 1.5.** The equation (1.1) has a rich structure being completely integrable, and the inverse scattering transform (IST) on the torus might reveal some fundamental differences compared to the real line case. In recent articles, Liu, Perry, and Sulem [27], analyze more closely the IST method on  $\mathbb{R}$  initiated by Lee [25] and prove global existence in a “spectrally determined” open subset of  $H^{2,2}(\mathbb{R}) := H^2(\mathbb{R}) \cap L^{2,m}(\mathbb{R})$  neighboring 0. Also via IST, Pelinovsky and Shimabukuro [36] construct unique global solutions for DNLS in  $H^2(\mathbb{R}) \cap \{f \in L^{2,1}(\mathbb{R}), f' \in L^{2,1}(\mathbb{R})\}$  with Lipschitz continuous dependence on the initial data. Here,  $L^{2,m}(\mathbb{R})$  denotes the weighted Lebesgue space  $L^2(\mathbb{R}; (1 + |x|^2)^{\frac{m}{2}} dx)$ .

**Remark 1.6.** In view of the local well-posedness result in the scale of Fourier-Lebesgue spaces by Grünrock and Herr [12], it would be interesting to investigate via the  $I$ -method the global dynamics of the DNLS flow in  $\mathcal{FL}^{\frac{1}{2},r}(\mathbb{T})$ , for the appropriate range in  $r$ , to complement the almost sure global well-posedness result of Nahmod, Oh, Rey-Bellet, and Staffilani [35]. This is also to be studied in the Euclidean case, where the local well-posedness was established by Grünrock in [11]. In the same direction of thought, we mention that recently, Takaoka [41] proved the existence of local  $H^s(\mathbb{T})$ -solutions with small (unquantified) mass in the range  $\frac{12}{25} < s < \frac{1}{2}$  by establishing a priori estimates for the gauge equivalent equation (1.18); also, his work [40] on the energy exchange behavior for a variant of (1.18) might provide further insight on the DNLS dynamics above the mass threshold  $4\pi$ .

As a summary, we outline the content of the present article. In Section 2, we introduce function spaces and review linear estimates (including a revised bilinear  $L^4$ -Strichartz estimate) that are used throughout the paper. After recalling the gauge transformation augmented with a translation operator, Theorem 1.1 is reduced to Proposition 2.14 concerning the global solutions of the periodic gauge equivalent equation (1.18). Also in Section 2, we provide the adaptation to the periodic setting of the Gagliardo-Nirenberg inequalities, and we introduce some further notation and constitutive elements for the  $I$ -method as close to previous implementations [7, 8, 10, 31] as possible. Then, we build up the  $I$ -method apparatus, beginning in Section 3. We first show that the coercivity property of the energy functional carries over to the periodic setting under the same mass condition (1.8) as in the Euclidean setting, and then we incorporate the momentum functional to obtain  $\dot{H}^1$ -norm control for solutions corresponding to the improved mass threshold (1.17). In Section 4, we provide a modified local well-posedness result based on existing local multi-linear estimates and an interpolation lemma for the  $I$ -operator. In this instantiation of the  $I$ -method scheme, we construct a third generation modified energy functional in Section 5 after revisiting the first and second generation energies, as well as discussing the frequency regions



that previously did not allow reaching the regularity  $s = \frac{1}{2}$ . In the same section, we also revisit the crafting of the resonant set from the real line setting and we provide pointwise bounds on multipliers which are used in the following two sections. Hence, in Section 6 we analyze the growth of the third generation modified energy and conclude with its almost conservation property, whereas in Section 7 we show that it stays close to the first generation modified energy. The almost conservation of the modified momentum follows similarly to the Euclidean case and is also established in Section 7. In Section 8, we modify the usual  $I$ -method argument to include the almost conserved momentum and we finish the proof of Proposition 2.14. Finally, the Appendix provides the proof of a technical lemma that immediately implies Proposition 1.2.

**Acknowledgements.** The author would like to thank his advisor, Tadahiro Oh, for suggesting this problem and for providing several fruitful ideas. He is also thankful to Yuzhao Wang, Soonsik Kwon, and Professor Yoshio Tsutsumi for their availability in discussing various aspects of this problem.

This work was supported by the UK's Engineering and Physical Sciences Research Council grant EP/L016508/1 and the Scottish Funding Council grant 242635599 for the University of Edinburgh and the Heriot-Watt University Centre for Doctoral Training, the Maxwell Institute Graduate School in Analysis and its Applications.

## 2. NOTATIONS AND BASIC ESTIMATES

A quantity of the form  $\alpha \pm$  is a shorthand for  $\alpha \pm \varepsilon$  with  $\alpha \in \mathbb{R}$  and  $\varepsilon > 0$  arbitrarily small (if two or more such quantities appear in the same relation, the dependence between  $\varepsilon$ 's is straightforward and can be ignored). By  $A \lesssim B$  ( $A \ll B$ ) we mean the existence of a large positive constant  $C$  such that  $A \leq CB$  ( $CA \leq B$ ); we write  $A \sim B$  if and only if  $A \lesssim B$  and  $B \lesssim A$ . Also, we write  $A = B + O(C)$  if and only if  $|A - B| \lesssim C$ . Throughout this work,  $\eta$  denotes a smooth time cut-off function with  $\eta \equiv 1$  on  $[-1, 1]$  and  $\eta \equiv 0$  outside  $(-2, 2)$ .

For reasons that are made clear in Section 8, we need to use the scaling transformation (1.5). Thus, we work on the parametrized torus  $\mathbb{T}_\lambda := \mathbb{R}/2\pi\lambda\mathbb{Z} \simeq [0, 2\pi\lambda)$ , and Fourier modes in  $\mathbb{Z}_\lambda := \frac{1}{\lambda}\mathbb{Z}$ . The convention we use for the (spatial) Fourier transform of a  $2\pi\lambda$ -periodic function is

$$\widehat{f}(k) = \int_0^{2\pi\lambda} e^{-ikx} f(x) dx \quad , \quad k \in \mathbb{Z}_\lambda$$

which is inverted by

$$\check{g}(x) = \frac{1}{2\pi\lambda} \sum_{k \in \mathbb{Z}_\lambda} e^{ikx} g(k) \quad , \quad x \in [0, 2\pi\lambda].$$

The convolution products on  $\mathbb{T}_\lambda$  and  $\mathbb{Z}_\lambda$  are given by

$$\begin{aligned} f * g(x) &= \int_0^{2\pi\lambda} f(x-y)g(y) dy, \\ a \star b(k) &= \frac{1}{2\pi\lambda} \sum_{h \in \mathbb{Z}_\lambda} a(k-h)b(h), \end{aligned}$$

respectively. We have  $\widehat{fg}(k) = \widehat{f} \star \widehat{g}(k)$ , and by endowing  $\mathbb{Z}_\lambda$  with the scaled counting measure  $(dk)_\lambda := \frac{1}{2\pi\lambda} d\#$ , the inner products on  $L^2(\mathbb{T}_\lambda)$  and  $L^2(\mathbb{Z}_\lambda)$  are

$$\begin{aligned}\langle f, g \rangle_{L^2(\mathbb{T}_\lambda)} &= \int_0^{2\pi\lambda} f(x) \overline{g(x)} dx, \\ \langle a, b \rangle_{L^2(\mathbb{Z}_\lambda)} &= \frac{1}{2\pi\lambda} \sum_{k \in \mathbb{Z}_\lambda} a(k) \overline{b(k)} = \int_{\mathbb{Z}_\lambda} a(k) \overline{b(k)} (dk)_\lambda,\end{aligned}$$

respectively. Then, the Parseval and Plancherel identities are written as

$$\begin{aligned}\langle f, \check{a} \rangle_{L^2(\mathbb{T}_\lambda)} &= \langle \widehat{f}, a \rangle_{L^2(\mathbb{Z}_\lambda)}, \\ \|f\|_{L^2(\mathbb{T}_\lambda)} &= \|\widehat{f}\|_{L^2(\mathbb{Z}_\lambda)}.\end{aligned}$$

The Sobolev space  $H^s(\mathbb{T}_\lambda)$ , respectively the Fourier Lebesgue space  $\mathcal{FL}^{s,r}(\mathbb{T}_\lambda)$  are the completion of the  $2\pi\lambda$ -periodic  $C^\infty$  functions with respect to the norms

$$\|f\|_{H^s(\mathbb{T}_\lambda)} := \|\langle k \rangle^s \widehat{f}(k)\|_{L^2(\mathbb{Z}_\lambda)}, \quad (2.1)$$

$$\|f\|_{\mathcal{FL}^{s,r}(\mathbb{T}_\lambda)} := \|\langle k \rangle^s \widehat{f}(k)\|_{L^r(\mathbb{Z}_\lambda)}, \quad (2.2)$$

where  $\langle k \rangle := (1 + |k|^2)^{\frac{1}{2}}$ ,  $k \in \mathbb{Z}_\lambda$ , for any  $s \geq 0$ ,  $r \geq 1$ . We also use the homogeneous Sobolev norm:

$$\|f\|_{\dot{H}^s(\mathbb{T}_\lambda)} := \| |k|^s \widehat{f}(k) \|_{L^2(\mathbb{Z}_\lambda)}.$$

**Remark 2.1.** Notice that for any  $k \neq 0$ , uniformly in  $\lambda \geq 1$ , we have

$$|k| \leq \langle k \rangle \lesssim \lambda |k|, \quad (2.3)$$

and thus, in the periodic setting,  $\langle k \rangle \sim 1$  for all  $|k| \lesssim \lambda$ .

By  $\mathcal{S}_\lambda$  we denote the class of functions  $u^\lambda : \mathbb{R} \times \mathbb{T}_\lambda \rightarrow \mathbb{C}$  which are Schwartz in  $t$ ,  $2\pi\lambda$ -periodic and  $C^\infty$  in  $x$ . With a slight abuse of notation, the time-space Fourier transform and its inverse are

$$\begin{aligned}\widehat{u}(\tau, k) &= \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} e^{-i(\tau t + kx)} u(t, x) dx dt, \quad \tau \in \mathbb{R}, k \in \mathbb{Z}_\lambda, \\ \check{v}(t, x) &= \int_{\mathbb{Z}_\lambda} \int_{\mathbb{R}} e^{i(\tau t + kx)} v(\tau, k) d\tau (dk)_\lambda, \quad t \in \mathbb{R}, x \in \mathbb{T}_\lambda.\end{aligned}$$

Nonlinear interactions take on the Fourier side the form

$$\begin{aligned}\widehat{uv}(\tau, k) &= \widehat{u} \star \widehat{v}(\tau, k) = \frac{1}{2\pi\lambda} \sum_{k_1 \in \mathbb{Z}_\lambda} \int_{\mathbb{R}} \widehat{u}(\tau_1, k_1) \widehat{v}(\tau - \tau_1, k - k_1) d\tau_1 \\ &= \int_{k_1 + k_2 = k} \int_{\tau_1 + \tau_2 = \tau} \widehat{u}(\tau_1, k_1) \widehat{v}(\tau_2, k_2) d\tau_1 (dk_1)_\lambda.\end{aligned}$$

The unitary group on  $L^2(\mathbb{T}_\lambda)$  determined by the linear Schrödinger equation on  $\mathbb{T}_\lambda$  is given by

$$(U_\lambda(t)f)(x) = \frac{1}{2\pi\lambda} \sum_{k \in \mathbb{Z}_\lambda} e^{ikx + itk^2} \widehat{f}(k). \quad (2.4)$$

For  $s, b \in \mathbb{R}$  (spatial and temporal regularity indices), we define the  $X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)$  space as the completion of  $\mathcal{S}_\lambda$  under the norm

$$\|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)} := \|\langle k \rangle^s \langle \tau + k^2 \rangle^b \widehat{u}(t, k)\|_{L_t^2 L_k^2(\mathbb{R} \times \mathbb{Z}_\lambda)}. \quad (2.5)$$

It is well known that the (continuous) embedding  $X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda) \subset C_t H_x^s(\mathbb{R} \times \mathbb{T}_\lambda)$  holds if and only if  $b > \frac{1}{2}$ . Since the trilinear estimate needed for the local well-posedness theory (see Lemma 4.2 below) holds only at  $b = \frac{1}{2}$ , we introduce the spaces  $Y^{s,b}$  and  $Z^s$  (with a slightly stronger norm than the  $X^{s,\frac{1}{2}}$ -norm) via the norms

$$\|u\|_{Y^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)} := \|\langle k \rangle^s \langle \tau + k^2 \rangle^b \widehat{u}(t, k)\|_{L_k^2 L_\tau^1(\mathbb{Z}_\lambda \times \mathbb{R})}, \quad (2.6)$$

$$\|u\|_{Z^s(\mathbb{R} \times \mathbb{T}_\lambda)} := \|u\|_{X^{s,\frac{1}{2}}(\mathbb{R} \times \mathbb{T}_\lambda)} + \|u\|_{Y^{s,0}(\mathbb{R} \times \mathbb{T}_\lambda)}, \quad (2.7)$$

and the companion space  $\widetilde{Z}^s$  by using

$$\|u\|_{\widetilde{Z}^s(\mathbb{R} \times \mathbb{T}_\lambda)} := \|u\|_{X^{s,-\frac{1}{2}}(\mathbb{R} \times \mathbb{T}_\lambda)} + \|u\|_{Y^{s,-1}(\mathbb{R} \times \mathbb{T}_\lambda)}. \quad (2.8)$$

We have  $Y^{s,0}(\mathbb{R} \times \mathbb{T}_\lambda) \subset C_t H_x^s(\mathbb{R} \times \mathbb{T}_\lambda)$  and therefore  $Z^s = X^{s,\frac{1}{2}} \cap Y^{s,0} \subset C_t H_x^s$ .

For a given time interval  $J$ , the time localized Fourier restriction norms are defined via

$$\|u\|_{X^{s,b}(J \times \mathbb{T}_\lambda)} := \inf\{\|v\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)} : v|_J = u\}, \quad (2.9)$$

and similarly for  $Y^{s,b}(J \times \mathbb{T}_\lambda)$ ,  $Z^s(J \times \mathbb{T}_\lambda)$ , and  $\widetilde{Z}^s(J \times \mathbb{T}_\lambda)$ .

By the Riemann-Lebesgue lemma and Hölder inequality, we have

$$\begin{aligned} \|u\|_{L_{t,x}^\infty(\mathbb{R} \times \mathbb{T}_\lambda)} &\lesssim \|\widehat{u}\|_{L_\tau^1(\mathbb{R})} \| \cdot \|_{L_k^1(\mathbb{Z}_\lambda)} \\ &\leq \left( \frac{1}{2\pi\lambda} \sum_{k \in \mathbb{Z}_\lambda} \langle k \rangle^{-1-} \right)^{\frac{1}{2}} \left( \frac{1}{2\pi\lambda} \sum_{k \in \mathbb{Z}_\lambda} \langle k \rangle^{1+} \|\widehat{u}(\tau, k)\|_{L_\tau^1(\mathbb{R})}^2 \right)^{\frac{1}{2}} \\ &\lesssim \|\langle k \rangle^{\frac{1}{2}+} \widehat{u}\|_{L_\tau^1(\mathbb{R})} \| \cdot \|_{L_k^2(\mathbb{Z}_\lambda)} \end{aligned}$$

and thus

$$\|u\|_{L_{t,x}^\infty(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \|u\|_{Y^{\frac{1}{2}+,0}(\mathbb{R} \times \mathbb{T}_\lambda)}. \quad (2.10)$$

Similarly, by Minkowski's integral inequality, Riemann-Lebesgue lemma and Plancherel's identity, one obtains

$$\|u\|_{L_t^\infty H_x^s(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \|u\|_{Y^{s,0}(\mathbb{R} \times \mathbb{T}_\lambda)}, \quad (2.11)$$

for any  $s \in \mathbb{R}$ .

Additionally, we have the following linear estimates.

**Lemma 2.2.** [17, Lemma 3.6] *Let  $s \in \mathbb{R}$ . There exists  $c > 0$  such that*

$$\|\eta(t)U_\lambda(t)f\|_{Z^s(\mathbb{R} \times \mathbb{T}_\lambda)} \leq c\|f\|_{H^s(\mathbb{T}_\lambda)} \quad (2.12)$$

$$\left\| \eta(t) \int_0^t U_\lambda(t-\tau)F(\tau, \cdot) d\tau \right\|_{Z^s(\mathbb{R} \times \mathbb{T}_\lambda)} \leq c\|F\|_{\widetilde{Z}^s(\mathbb{R} \times \mathbb{T}_\lambda)} \quad (2.13)$$

for all  $f \in H^s$  and all  $F \in \mathcal{S}_\lambda$ .

**Lemma 2.3.** *Let  $2 \leq p, q < \infty$ ,  $b \geq \frac{1}{2} - \frac{1}{p}$ ,  $s \geq \frac{1}{2} - \frac{1}{q}$ ,  $\lambda \geq 1$ . For  $u \in \mathcal{S}_\lambda$ , we have*

(1) *Sobolev estimates:*

$$\|u\|_{L_t^p H_x^s(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)}, \quad (2.14)$$

$$\|u\|_{L_t^\infty H_x^s(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \|u\|_{X^{s, \frac{1}{2}+}(\mathbb{R} \times \mathbb{T}_\lambda)}, \quad (2.15)$$

$$\|u\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)}, \quad (2.16)$$

$$\|u\|_{L_t^\infty L_x^\infty(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \|u\|_{X^{\frac{1}{2}+, \frac{1}{2}+}(\mathbb{R} \times \mathbb{T}_\lambda)}; \quad (2.17)$$

(2) *Strichartz estimates:*

$$\|u\|_{L_{t,x}^4(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \|u\|_{X^{0, \frac{3}{8}}(\mathbb{R} \times \mathbb{T}_\lambda)}, \quad (2.18)$$

$$\|u\|_{L_{t,x}^6(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \lambda^{0+} \|u\|_{X^{0+, \frac{1}{2}+}(\mathbb{R} \times \mathbb{T}_\lambda)} \quad (2.19)$$

with implicit constants independent of  $\lambda \geq 1$ .

One can prove the first part by using the interaction representation

$$\|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)} = \|U_\lambda(-t)u(t, x)\|_{H_x^s H_t^b(\mathbb{T}_\lambda \times \mathbb{R})}, \quad (2.20)$$

the classical Sobolev inequalities, Minkowski's integral inequality and the fact that the operators  $U_\lambda(t)$  are unitary on  $H_x^s(\mathbb{T}_\lambda)$ . The second part can be justified by going over the Strichartz estimates due to Bourgain [3] and revisiting the counting arguments, but now accounting for Fourier modes in  $\mathbb{Z}_\lambda$  rather than  $\mathbb{Z}$  (e.g. there are  $O(\lambda M)$  elements  $k$  in  $\mathbb{Z}_\lambda$  satisfying  $|k| \lesssim M$ , there is a normalizing factor in the measure placed on  $\mathbb{Z}_\lambda$ , etc.). It turns out that the  $L^4$ -Strichartz estimate has an implicit constant independent of  $\lambda$ , while the  $L^6$ -Strichartz estimate has a logarithmic loss in  $\lambda$  (in addition to the loss in derivative).

By interpolating the Strichartz estimate (2.19) with the Sobolev inequality (2.16) (for  $p = q = 6$ ), we also have

$$\|u\|_{L_{t,x}^6(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \lambda^{0+} \|u\|_{X^{0+, \frac{1}{2}-}(\mathbb{R} \times \mathbb{T}_\lambda)}. \quad (2.21)$$

We note that the estimates (2.14)-(2.21) also hold for Fourier restriction norms on a time interval  $J$  rather than on the entire real line.

We record the following scaling properties of the space-time norms introduced above when using (1.5) and a parameter  $\lambda \geq 1$ :

$$\|u^\lambda\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{T}_\lambda)} = \lambda^{\frac{2}{p} + \frac{1}{q} - \frac{1}{2}} \|u\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{T})},$$

$$\|u^\lambda\|_{L_t^p \dot{H}_x^s(\mathbb{R} \times \mathbb{T}_\lambda)} = \lambda^{-s + \frac{2}{p}} \|u\|_{L_t^p \dot{H}_x^s(\mathbb{R} \times \mathbb{T})},$$

and

$$\lambda^{-s + \frac{2}{p}} \|u\|_{L_t^p H_x^s(\mathbb{R} \times \mathbb{T})} \lesssim \|u^\lambda\|_{L_t^p H_x^s(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \lambda^{\frac{2}{p}} \|u\|_{L_t^p H_x^s(\mathbb{R} \times \mathbb{T})}.$$

For  $s, b \geq 0$ , we have

$$\lambda^{-1} \|u^\lambda\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{T})} \lesssim \lambda^{-1+s+2b} \|u^\lambda\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)},$$

while for  $s \geq 0, b < 0$ , we record

$$\lambda^{-1+2b} \|u^\lambda\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{T})} \lesssim \lambda^{-1+s} \|u^\lambda\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}_\lambda)}.$$

We also use the following lemma when dealing with sharp time-cutoff functions:

**Lemma 2.4.** *Let  $s \in \mathbb{R}$  and suppose  $\phi \in H_t^{\frac{1}{2}-}(\mathbb{R})$ . Then:*

$$\|\phi u\|_{X^{s, \frac{1}{2}-}(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \|\phi\|_{H_t^{\frac{1}{2}-}(\mathbb{R})} \|u\|_{X^{s, \frac{1}{2}}(\mathbb{R} \times \mathbb{T}_\lambda)}.$$

*Proof.* By (2.20),

$$\|\phi u\|_{X^{s, b'}(\mathbb{R} \times \mathbb{T}_\lambda)} = \|\phi(t)U_\lambda(-t)u(t, x)\|_{H_x^s H_t^{b'}(\mathbb{T}_\lambda \times \mathbb{R})}$$

and let  $J_t := \langle \partial_t \rangle$ . Then, via the fractional Leibniz rule, we have

$$\|\phi(t)U_\lambda(-t)u(t)\|_{H_t^{b'}} \lesssim \|J_t^{b'} \phi\|_{L_t^p} \|U_\lambda(-t)u(t)\|_{L_t^q} + \|\phi\|_{L_t^q} \|J_t^{b'}(U_\lambda(-t)u(t))\|_{L_t^p}, \quad (2.22)$$

where  $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ . We take  $b' := \frac{1}{2} - < b < \frac{1}{2}$  and  $p > 2$  so that we have the continuous Sobolev embedding  $H_t^{b-b'}(\mathbb{R}) \subset L_t^p(\mathbb{R})$ . Consequently, we also have the Sobolev embedding  $H_t^b(\mathbb{R}) \subset L_t^q(\mathbb{R})$ . Then, the conclusion follows from (2.22) and triangle inequality for the  $H_x^s(\mathbb{R})$ -norm.  $\square$

**2.1. A bilinear  $L^4$ -Strichartz estimate.** The following result is a key ingredient in the analysis of the almost conservation estimates as it is a refinement of the  $L^4$ -Strichartz estimate that provides a decaying factor in  $\lambda$ . Such an estimate is similar to the bilinear  $L^4$ -estimate in the non-periodic setting [7, Lemma 7.1], and we point out that for  $\lambda \rightarrow \infty$ , we recover the same decay rate. For Schrödinger evolutions on the one-dimensional torus, this estimate (but without pointing out the alternative (ii)) was first proved in [10].

**Lemma 2.5.** *Let  $\lambda \geq 1$ ,  $N_1, N_2 \in 2^{\mathbb{Z}}$  and suppose  $\phi_1, \phi_2$  are smooth functions on  $\mathbb{T}_\lambda$  with  $\text{supp}(\widehat{\phi_j}) \subset \{k \in \mathbb{Z}_\lambda : |k| \sim N_j\}$ ,  $j = 1, 2$ . Assume that either*

(i)  $N_1 \gg N_2$ , or

(ii)  $N_1 \sim N_2$  and  $k_1 k_2 < 0$  for all  $k_1 \in \text{supp}(\widehat{\phi_1})$ ,  $k_2 \in \text{supp}(\widehat{\phi_2})$ .

Then

$$\|(\eta(t)U_\lambda(t)\phi_1)(\eta(t)U_\lambda(t)\phi_2)\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim C(\lambda, N_1) \|\phi_1\|_{L_x^2(\mathbb{T}_\lambda)} \|\phi_2\|_{L_x^2(\mathbb{T}_\lambda)} \quad (2.23)$$

where

$$C(\lambda, N_1) = \begin{cases} 1 & , \text{ if } N_1 \lesssim 1 \\ (\frac{1}{\lambda} + \frac{1}{N_1})^{\frac{1}{2}} & , \text{ if } N_1 \gg 1 \end{cases}. \quad (2.24)$$

Moreover, suppose  $u_1, u_2 \in \mathcal{S}_\lambda$  are Fourier supported in  $\{|k_1| \sim N_1\}$  and  $\{|k_2| \sim N_2\}$ , respectively, for all times  $t$ . Then, under the same assumption on the two frequency supports, we have

$$\|u_1 u_2\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim_\varepsilon C(\lambda, N_1)^{1-2\varepsilon} \|u_1\|_{X^{0, \frac{1}{2}-\varepsilon}(\mathbb{R} \times \mathbb{T}_\lambda)} \|u_2\|_{X^{0, \frac{1}{2}-\varepsilon}(\mathbb{R} \times \mathbb{T}_\lambda)}, \quad (2.25)$$

for any  $\varepsilon > 0$  sufficiently small.

**Remark 2.6.** In [49, Proposition 2.1], there seems to be a mistake in the case  $N_1 \sim N_2$ : the two Fourier supports should be localized on opposite sides of the origin on the real line in order for (2.23) to be true. The estimate with this additional assumption was used in proving Cases (2) and (3) of [49, Lemma 7.5]. Although with similar ideas as in the proof of [10, Proposition 3.7], we decided to present the proof so that this observation becomes clear.

*Proof.* By Plancherel's identity, the left hand side of (2.23) becomes

$$\left\| \int_{\tau_1 + \tau_2 = \tau} \int_{k_1 + k_2 = k} \widehat{\eta}(\tau_1 + k_1^2) \widehat{\eta}(\tau_2 + k_2^2) \widehat{\phi}_1(k_1) \widehat{\phi}_2(k_2) (dk_1)_\lambda d\tau_1 \right\|_{L_\tau^2 L_k^2}.$$

We denote  $\psi := \widehat{\eta} * \widehat{\eta}$ , and without loss of generality, we can assume that  $\psi$  is  $\mathbb{R}$ -valued and non-negative.<sup>8</sup> Then

$$\int_{\mathbb{R}} \widehat{\eta}(\tau_1 + k_1^2) \widehat{\eta}(\tau - \tau_1 + k_2^2) d\tau_1 = \psi(\tau + k_1^2 + k_2^2) \geq 0$$

and by Hölder's inequality, we have

$$\begin{aligned} & \left\| \int_{k_1 + k_2 = k} \psi(\tau + k_1^2 + k_2^2) \widehat{\phi}_1(k_1) \widehat{\phi}_2(k_2) (dk_1)_\lambda \right\|_{L_\tau^2 L_k^2} \\ & \leq \left\| \left( \int_{k_1 + k_2 = k} \psi(\tau + k_1^2 + k_2^2) (dk_1)_\lambda \right)^{\frac{1}{2}} \right. \\ & \quad \times \left. \left( \int_{k_1 + k_2 = k} \psi(\tau + k_1^2 + k_2^2) |\widehat{\phi}_1(k_1)|^2 |\widehat{\phi}_2(k_2)|^2 (dk_1)_\lambda \right)^{\frac{1}{2}} \right\|_{L_\tau^2 L_k^2} \\ & \leq M \left( \int_{\mathbb{Z}_\lambda} \int_{\mathbb{Z}_\lambda} \int_{\mathbb{R}} \psi(\tau + k_1^2 + k_2^2) |\widehat{\phi}_1(k_1)|^2 |\widehat{\phi}_2(k_2)|^2 d\tau (dk_1)_\lambda (dk)_\lambda \right)^{\frac{1}{2}} \\ & \leq M \|\psi\|_{L^1(\mathbb{R})}^{\frac{1}{2}} \|\phi_1\|_{L^2(\mathbb{T}_\lambda)} \|\phi_2\|_{L^2(\mathbb{T}_\lambda)}, \end{aligned}$$

where we applied Fubini's theorem and we denoted

$$M := \left( \sup_{k, \tau} \int_{k_1 + k_2 = k} \psi(\tau + k_1^2 + k_2^2) (dk_1)_\lambda \right)^{\frac{1}{2}}.$$

Thus, in order to obtain (2.23), it remains to show that  $M \lesssim C(\lambda, N_1)$ .

Since  $\psi$  is a Schwartz function, it is rapidly decaying, and so we can split  $\mathbb{R}$  into disjoint intervals  $I_j$  ( $j \in \mathbb{Z}$ )<sup>9</sup> such that for all  $j$  we have  $|I_j| \sim 1$  and  $\|\psi|_{I_j}\|_{L^\infty} \lesssim 2^{-|j|}$ . Given  $k \in \mathbb{Z}_\lambda$ ,  $\tau \in \mathbb{R}$ , and  $j \in \mathbb{Z}$ , we consider the set

$$S_{k, \tau, j} = \{k_1 \in \mathbb{Z}_\lambda : k_1 \in \text{supp}(\widehat{\phi}_1), k - k_1 \in \text{supp}(\widehat{\phi}_2), \tau + k_1^2 + (k - k_1)^2 \in I_j\}$$

and we estimate

$$M \lesssim \left( \sup_{k, \tau} \sum_{j \in \mathbb{Z}} \left( \frac{1}{\lambda} \# S_{k, \tau, j} \right) 2^{-|j|} \right)^{\frac{1}{2}},$$

where  $\# S_{k, \tau, j}$  denotes the cardinality of  $S_{k, \tau, j}$ .

If  $N_1 \lesssim 1$ , then clearly

$$\# S_{k, \tau, j} \leq \# \left\{ k_1 \in \frac{1}{\lambda} \mathbb{Z} : |k_1| \lesssim 1 \right\} \lesssim \lambda$$

and thus  $M \lesssim 1$ .

8. In general, we can write  $\psi = \psi_+ - \psi_- + i\psi^+ - i\psi^-$  with the four components satisfying the non-negativity assumption, from where we can carry on analogous arguments for each of these terms.

9. If  $\psi$  were compactly supported, it would be enough to consider only one such interval, namely a finite-length interval which includes the support of  $\psi$ .

Now let us assume  $N_1 \gg 1$ . To estimate the cardinality of a nonempty set  $S_{k,\tau,j}$ , we denote

$$f_{k,\tau}(k_1) := \tau + k_1^2 + (k - k_1)^2.$$

Notice that

$$|f'_{k,\tau}(k_1)| = 2|k_1 - (k - k_1)| \sim N_1, \quad (2.26)$$

and that this property holds not only when  $N_1 \gg N_2$  but also when  $k_1$  and  $k - k_1$  have opposite signs, and this is ensured by assumption (ii). From (2.26) and the mean value theorem, we get that

$$\#S_{k,\tau,j} \lesssim 1 + \frac{\lambda}{N_1},$$

uniformly in  $j$  (if  $\lambda \lesssim N_1$  there might be only one element in  $S_{k,\tau,j}$ ).

For the last part, by the transference principle for  $X^{s,b}$  spaces (see for example [43, Lemma 2.9]), the estimate (2.23) implies

$$\|u_1 u_2\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim C(\lambda, N_1) \|u_1\|_{X^{0, \frac{1}{2}+}(\mathbb{R} \times \mathbb{T}_\lambda)} \|u_2\|_{X^{0, \frac{1}{2}+}(\mathbb{R} \times \mathbb{T}_\lambda)}. \quad (2.27)$$

On the other hand, by Hölder inequality and the  $L^4$ -Strichartz estimate, we have

$$\|u_1 u_2\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \|u_1\|_{X^{0, \frac{3}{8}}(\mathbb{R} \times \mathbb{T}_\lambda)} \|u_2\|_{X^{0, \frac{3}{8}}(\mathbb{R} \times \mathbb{T}_\lambda)}. \quad (2.28)$$

By interpolating (2.27) and (2.28), we obtain (2.25) for  $\varepsilon > 0$  sufficiently small.  $\square$

**Remark 2.7.** We point out that the implicit constant in (2.25) depends on  $\varepsilon$ . Hence, we cannot disregard the logarithmic loss in the constant  $C(\lambda, N_1)$ . This loss is essentially the reason for which we need to introduce the second correction term in (5.25) in the third iteration of the  $I$ -method (see also Remark 5.12).

**Remark 2.8.** Notice that, under assumption (i) of the above lemma, the estimate (2.25) holds if we replace one of the functions on the left hand side with its conjugate (or equivalently, one of the  $X^{0, \frac{1}{2}}$ -norms in the right hand side with the  $\overline{X}^{0, \frac{1}{2}}$ -norm as defined in [17]). This is no longer true under assumption (ii).

We use the above bilinear estimate essentially in the regime  $1 \leq \lambda \lesssim N_1$ , and thus, in our estimates,  $C(\lambda, N_1) \sim \lambda^{-\frac{1}{2}}$ .

**2.2. Gagliardo-Nirenberg inequalities in the periodic setting.** We recall that on the real line, we have the sharp Gagliardo-Nirenberg inequalities

$$\|f\|_{L^6(\mathbb{R})} \leq \left(\frac{2}{\pi}\right)^{\frac{1}{3}} \|\partial_x f\|_{L^2(\mathbb{R})}^{\frac{1}{3}} \|f\|_{L^2(\mathbb{R})}^{\frac{2}{3}}, \quad (2.29)$$

$$\|f\|_{L^6(\mathbb{R})} \leq C_{\text{GN}} \|\partial_x f\|_{L^2(\mathbb{R})}^{\frac{1}{9}} \|f\|_{L^4(\mathbb{R})}^{\frac{8}{9}}, \quad (2.30)$$

where  $C_{\text{GN}} := 3^{\frac{1}{6}}(2\pi)^{-\frac{1}{9}}$ . For (2.29) we refer to [47], whereas for (2.30), see [1].

**Remark 2.9.** Extremising functions in (2.29) are given by solutions of

$$-f'' + f - \frac{3}{16}f^5 = 0.$$

Up to translations, there is a unique positive, decaying solution, called the ground state and denoted by  $Q$ , for which we have the Pohozaev identities  $M[Q] = 2\pi$  and  $E[Q] = 0$ . We also know that for any  $u_0 \in H^1(\mathbb{R})$  with  $M[u_0] < M[Q]$ , we have  $E[u_0] > 0$ . Such a  $Q$  is also a stationary solution of (1.18) on  $\mathbb{R}$  with  $\beta = \frac{3}{4}$ . Moreover, there exists  $u_0 \in H^1(\mathbb{R})$  with  $2\pi < M[u_0] < 2\pi + \varepsilon$  such that  $E[u_0] < 0$ .

On the other hand, solutions of

$$-f'' + f^3 - \frac{3}{16}f^5 = 0$$

achieve the optimal constant  $C_{GN}$  in (2.30). The ground state solution  $W$  has  $M[W] = 4\pi$ .

On  $\mathbb{T}$ , inequalities of the above form cannot hold, simply for the fact that constant functions provide counterexamples. However, the situation is similar to the Poincaré inequality, and in fact, using elementary arguments,<sup>10</sup> it was shown in [17, Appendix C] the following inequality:

$$\|(|f|^2 - \mu[f])f\|_{L^2(\mathbb{T}_\lambda)} \leq \|\partial_x f\|_{L^2(\mathbb{T}_\lambda)} \|f\|_{L^2(\mathbb{T}_\lambda)}^2, \quad (2.31)$$

for any  $2\pi\lambda$ -periodic function  $f$ . Although (2.31) can be used to study the coercivity of  $E$  (see Lemma 3.2 below), we use here the following result (see e.g. Lebowitz, Rose and Speer [24, Lemma 4.1]) since it yields the same mass threshold  $M[u_0] < 2\pi$  as in the Euclidean setting.

**Lemma 2.10.** *For any  $\varepsilon > 0$ , there exists a constant  $K_\varepsilon > 0$  (independent of  $\lambda$ ) such that*

$$\|f\|_{L^6(\mathbb{T}_\lambda)}^6 \leq \left(\frac{4}{\pi^2} + \varepsilon\right) \|\partial_x f\|_{L^2(\mathbb{T}_\lambda)}^2 \|f\|_{L^2(\mathbb{T}_\lambda)}^4 + K_\varepsilon \|f\|_{L^2(\mathbb{T}_\lambda)}^6, \quad (2.32)$$

for all  $f \in H^1(\mathbb{T}_\lambda)$ .

With similar arguments, one can adapt (2.30) to  $\mathbb{T}_\lambda$  as well.

**Lemma 2.11.** *Let  $\delta > 0$ . We have*

$$\|f\|_{L^6(\mathbb{T}_\lambda)} \leq C_{GN} \left(1 + \frac{\delta}{5\pi\lambda}\right)^{\frac{2}{9}} \left(\|\partial_x f\|_{L^2(\mathbb{T}_\lambda)}^2 + \frac{1}{\pi\lambda\delta} \|f\|_{L^2(\mathbb{T}_\lambda)}^2\right)^{\frac{1}{18}} \|f\|_{L^4(\mathbb{T}_\lambda)}^{\frac{8}{9}} \quad (2.33)$$

for all  $f \in H^1(\mathbb{T}_\lambda)$ .

The proof is a slight modification of [34, Lemma 2.2] which used the  $L^4$ -norm rather than the  $L^2$ -norm of  $f$  in the first factor on the right hand side above.

**2.3. The gauge transformation on  $\mathbb{T}_\lambda$ .** Following Herr [17, 18], we consider

$$\mathcal{G}_\beta : L_x^2(\mathbb{T}_\lambda) \rightarrow L_x^2(\mathbb{T}_\lambda) \quad , \quad \mathcal{G}_\beta(f)(x) := e^{-i\beta\mathcal{J}(f)(x)} f(x) \quad , \quad (2.34)$$

where  $\mathcal{J}(f)$  is the mean-zero antiderivative of  $|f|^2 - \mu(f)$ , i.e.

$$\mathcal{J}(f)(x) := \frac{1}{2\pi\lambda} \int_0^{2\pi\lambda} \int_\theta^x |f(y)|^2 - \mu[f] dy d\theta \quad (2.35)$$

and

$$\mu[f] := \frac{1}{2\pi\lambda} \|f\|_{L^2(\mathbb{T}_\lambda)}^2. \quad (2.36)$$

Note that  $|\mathcal{G}_\beta f| = |f|$  and therefore  $\mu[\mathcal{G}_\beta f] = \mu[f]$ ; moreover  $\mathcal{G}_\beta$  is inverted by  $\mathcal{G}_{-\beta}$ .

Setting  $w(t, x) = \mathcal{G}_\beta(u(t))(x)$ , the derivative nonlinear Schrödinger equation (1.1) becomes

$$\begin{aligned} i\partial_t w + \partial_x^2 w - 2i\beta\mu[w] \partial_x w &= 2i(1 - \beta)|w|^2 \partial_x w + i(1 - 2\beta)w^2 \partial_x \bar{w} + \beta\mu[w]|w|^2 w \\ &\quad + \left(\frac{\beta}{2} - \beta^2\right)|w|^4 w - \psi[w]w, \end{aligned} \quad (2.37)$$

<sup>10</sup>. Strictly speaking, (2.31) was proved on  $\mathbb{T}$ , but it is also true on  $\mathbb{T}_\lambda$  as the inequality is scale invariant. The same result can be obtained by using the pointwise Poincaré inequality followed by an application of the Hölder inequality.



where

$$\psi[w] := \frac{\beta}{2\pi\lambda} \int_{\mathbb{T}_\lambda} \left( 2\operatorname{Im}(w\bar{w}_x) + \left(\frac{3}{2} - 2\beta\right)|w|^4 \right) dx + \beta^2\mu[w]^2. \quad (2.38)$$

Correspondingly, the momentum and energy functionals are

$$P[\mathcal{G}_{-\beta}(w)] = \int_{\mathbb{T}_\lambda} \left( \operatorname{Im}(w\bar{w}_x) + \left(\frac{1}{2} - \beta\right)|w|^4 \right) dx + \beta\mu[w]M[w] =: P_\beta[w], \quad (2.39)$$

$$\begin{aligned} E[\mathcal{G}_{-\beta}(w)] &= \int_{\mathbb{T}_\lambda} \left( |w_x|^2 + \left(\frac{3}{2} - 2\beta\right)|w|^2 \operatorname{Im}(w\bar{w}_x) + \left(\beta^2 - \frac{3}{2}\beta + \frac{1}{2}\right)|w|^6 \right) dx \\ &\quad + \frac{\beta}{2}\mu[w]\|w\|_{L_x^4}^4 + 2\beta\mu[w]P_\beta[w] - \beta^2\mu[w]^2M[w] =: E_\beta[w]. \end{aligned} \quad (2.40)$$

We point out that in the periodic setting, the terms coupled with  $\mu[w]$  and  $\psi[w]$  are new terms when comparing (2.37) to the Euclidean setting.

We can eliminate the auxiliary linear term on the left hand side of (2.37) by the translation transformation

$$w(t, x) \mapsto v(t, x + 2\beta\mu[w]t). \quad (2.41)$$

Correspondingly, we introduce the gauge transformation of spacetime functions

$$\mathcal{G}^\beta : C_t^0 L_x^2(J \times \mathbb{T}_\lambda) \rightarrow C_t^0 L_x^2(J \times \mathbb{T}_\lambda), \quad \mathcal{G}^\beta(u)(t, x) := \mathcal{G}_\beta(u(t))(x - 2\beta\mu[u(t)]t). \quad (2.42)$$

For the local well-posedness theory, it is necessary to use the gauge parameter  $\beta = 1$  so that the “bad” nonlinear term  $|w|^2\partial_x w$  in (2.37) is eliminated. Hence, in the sequel, we consider the equation on  $\mathbb{T}_\lambda$  corresponding to this gauge choice, namely

$$i\partial_t v + \partial_x^2 v = -iv^2\partial_x \bar{v} - \frac{1}{2}|v|^4 v + \mu[v]|v|^2 v - \psi[v]v, \quad (2.43)$$

where we recall that  $\mu[v] = \frac{1}{2\pi\lambda}\|v\|_{L^2(\mathbb{T}_\lambda)}^2$  and

$$\psi[v] := \frac{1}{2\pi\lambda} \int_{\mathbb{T}_\lambda} \left( 2\operatorname{Im}(v\bar{v}_x) - \frac{1}{2}|v|^4 \right) dx + \mu[v]^2. \quad (2.44)$$

**Remark 2.12.** The nonlinearity of (2.43) can be written in the form  $\mathcal{N} = \tilde{\mathcal{T}} + \tilde{\mathcal{Q}}$  by grouping terms as follows:

$$\tilde{\mathcal{T}}(v) := -i \left( v\bar{v}_x - 2i \oint_{\mathbb{T}_\lambda} \operatorname{Im}(v\bar{v}_x) dx \right) v, \quad (2.45)$$

$$\tilde{\mathcal{Q}}(v) := -\frac{1}{2} \left( |v|^4 - \oint_{\mathbb{T}_\lambda} |v|^4 dx \right) + \oint_{\mathbb{T}_\lambda} |v|^2 dx \left( |v|^2 - \oint_{\mathbb{T}_\lambda} |v|^2 dx \right) v. \quad (2.46)$$

An important observation here is that, on the Fourier side, by using the inclusion-exclusion principle we can write

$$\widehat{\tilde{\mathcal{T}}(v)}(k) = \frac{1}{(2\pi\lambda)^2} \sum_{\substack{k=k_{123} \\ k \neq k_1, k_3}} k_2 \widehat{v}(k_1) \widehat{v}(k_2) \widehat{v}(k_3) - \frac{1}{(2\pi\lambda)^2} k \widehat{v}(k) \widehat{v}(-k) \widehat{v}(k). \quad (2.47)$$

It was made clear in [12] that the above frequency cancelations are essential in establishing the estimate that deals with the derivative-cubic nonlinearity in the scale of Fourier-Lebesgue spaces  $\mathcal{FL}^{s,r}(\mathbb{T})$ . However, in Sobolev spaces  $H^s(\mathbb{T})$ , for  $s \geq \frac{1}{2}$ , one can handle the cubic-derivative term  $u^2\partial_x \bar{u}$  without the frequency cancelations (see Lemma 4.2 below due to Herr [17]). In (2.46), the coefficients of the subtracted terms do not allow symmetrization, hence we do not have useful frequency cancelations.

The following lemma provides the continuity properties of the gauge transformation that allow to transfer well-posedness results between various versions of (1.18), including (1.1) itself. We note that in order to have the Lipschitz continuity of  $\mathcal{G}^\beta$  (rather than of  $\mathcal{G}_\beta$ ) one needs to fix the  $L^2$ -norm of the functions at all times  $t$ .

**Lemma 2.13.** [17, Lemma 2.3] *Let  $s, r, \mu_0 \geq 0$ ,  $T > 0$ . There exists  $c = c(r, s, \lambda) > 0$  such that:*

$$(1) \text{ If } f, g \in B_r := \{f \in H^s(\mathbb{T}_\lambda) : \|f\|_{H^s(\mathbb{T}_\lambda)} \leq r\}, \text{ then}$$

$$\|\mathcal{G}_\beta(f) - \mathcal{G}_\beta(g)\|_{H^s(\mathbb{T}_\lambda)} \leq c\|f - g\|_{H^s(\mathbb{T}_\lambda)}. \quad (2.48)$$

$$(2) \text{ If } u, v \in B^{r, \mu_0}, \text{ where}$$

$$B^{r, \mu_0} := \{u \in C([-T, T]; H^s(\mathbb{T}_\lambda)) : \|u\|_{L_t^\infty H_x^s} \leq r, \mu[u(t)] = \mu_0 \text{ for all } t \in [-T, T]\},$$

then

$$\|\mathcal{G}^\beta(u)(t) - \mathcal{G}^\beta(v)(t)\|_{H^s(\mathbb{T}_\lambda)} \leq c\|u(t) - v(t)\|_{H^s(\mathbb{T}_\lambda)} \quad (2.49)$$

for all  $t \in [-T, T]$ .

We recall that the local well-posedness theory for (2.43) via a fixed point argument in the space  $Z^1$  was developed in [17, 18] (see the estimates in Lemma 4.2 below). Therefore, in order to get Theorem 1.1, we aim to prove that the  $H^s$ -solutions of (2.43) exist globally in time in the following sense:

**Proposition 2.14.** *Let  $\frac{1}{2} \leq s < 1$  and  $v_0 \in H^s(\mathbb{T})$  with  $M[v_0] < 4\pi$ . Then for any  $\varepsilon > 0$ , there exists  $c = c(\|v_0\|_{H^s(\mathbb{T})}, M[v_0], \varepsilon) < \infty$  such that for all  $T > 0$ , the solution  $v$  of (2.43) with  $v(0) = v_0$  satisfies*

$$\sup_{0 \leq t \leq T} \|v(t)\|_{H_x^s(\mathbb{T})} \leq c(1 + T)^{2-2s+\varepsilon}.$$

Since the equation (2.43) has the time reversibility symmetry  $v(t, x) \mapsto \overline{v(-t, -x)}$  and the  $L_x^2$ -norm is conserved along the evolution, the above result implies that the  $H_x^s$ -norm of any solution  $v$  of (2.43) does not blow up in finite time.

**2.4. The  $I$ -operator.** For  $0 \leq s < 1$  and  $N \gg 1$  a fixed dyadic number, we define the Fourier multiplication operator<sup>11</sup>

$$I : H^s(\mathbb{T}_\lambda) \rightarrow H^1(\mathbb{T}_\lambda), \quad \widehat{If}(k) = m(k)\widehat{f}(k), \quad k \in \mathbb{Z}_\lambda \quad (2.50)$$

where  $m : \mathbb{R} \rightarrow (0, 1]$  is an even, smooth, non-increasing function on  $[0, \infty)$ , chosen such that

$$m(\xi) = \begin{cases} 1 & , \text{ if } |\xi| \leq N \\ \left(\frac{N}{|\xi|}\right)^{1-s} & , \text{ if } |\xi| \geq 2N \end{cases}$$

and a smooth interpolant for  $N \leq |\xi| \leq 2N$ . Furthermore, for any  $s \geq \frac{1}{2}$ , the Fourier multiplier  $m(\cdot)$  can be chosen such that it satisfies the monotonicity property

$$\xi \mapsto m(\xi)\xi^{\frac{1}{2}} \text{ is non-decreasing on } [0, \infty). \quad (2.51)$$

11. The operator  $I$  depends on the regularity index  $s$  and the parameters  $N$  and  $\lambda$ , but we choose to omit them as indices of  $I$  whenever possible. However, in Lemma 4.4 it becomes necessary to point them out explicitly.

One easily checks that, for any  $0 \leq \theta < 1$ , we have

$$m(k)\langle k \rangle^{1-\theta} \gtrsim \begin{cases} N^{1-\theta} & , \text{ if } |k| \gg N \\ 1 & , \text{ if } |k| \lesssim N \end{cases} \quad (2.52)$$

in the regularity range  $\theta \leq s < 1$ , with implicit constants independent of  $\lambda$ .

We note that  $I$  behaves like the identity operator on frequencies smaller than  $N$  and integrates of order  $1-s$  on frequencies much bigger than  $N$ . Indeed,

$$\sum_{k \lesssim N} \langle k \rangle^{2s} |\widehat{u}(k)|^2 \lesssim \sum_{k \lesssim N} \langle k \rangle^2 m(k)^2 |\widehat{u}(k)|^2 \lesssim N^{2(1-s)} \sum_{k \lesssim N} \left( \frac{\langle k \rangle}{N} \right)^{2-2s} \langle k \rangle^{2s} m(k)^2 |\widehat{u}(k)|^2$$

and

$$\sum_{k \gg N} \langle k \rangle^2 \frac{1}{\langle k \rangle^{2-2s}} |\widehat{u}(k)|^2 \lesssim \sum_{k \gg N} \langle k \rangle^2 m(k)^2 |\widehat{u}(k)|^2 \lesssim \sum_{k \gg N} \langle k \rangle^2 \left( \frac{N}{\langle k \rangle} \right)^{2-2s} |\widehat{u}(k)|^2.$$

Therefore, we have

$$\|u\|_{H^s(\mathbb{T}_\lambda)} \lesssim \|Iu\|_{H^1(\mathbb{T}_\lambda)} \lesssim N^{1-s} \|u\|_{H^s(\mathbb{T}_\lambda)}, \quad (2.53)$$

as well as

$$\|u\|_{\dot{H}^s(\mathbb{T}_\lambda)} \lesssim \|Iu\|_{\dot{H}^1(\mathbb{T}_\lambda)} \lesssim N^{1-s} \|u\|_{\dot{H}^s(\mathbb{T}_\lambda)}. \quad (2.54)$$

**2.5. Multilinear forms.** As in [7, 8, 10, 31], we use the shorthand notations  $k_{12\dots n} := k_1 + k_2 + \dots + k_n$ ,  $k_{1-2} := k_1 - k_2$ , etc., as well as  $m_j := m(k_j)$ ,  $m_{jh} := m(k_{jh})$ , etc. Also, we set

$$\begin{aligned} \Gamma_n(\mathbb{T}_\lambda) &:= \{\mathbf{k} = (k_1, \dots, k_n) \in (\mathbb{Z}_\lambda)^n : k_{12\dots n} = 0\}, \\ \Gamma_n(\mathbb{R}) &:= \{(\tau_1, \dots, \tau_n) \in \mathbb{R}^n : \tau_1 + \dots + \tau_n = 0\} \end{aligned}$$

and we endow them with the measure induced from the scaled counting measure  $\frac{1}{(2\pi\lambda)^{n-1}} d\#$  and, respectively, from the Lebesgue measure  $d\tau_1 \dots d\tau_{n-1}$ , by pushing forward under the map  $(x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, -x_1 - \dots - x_{n-1})$ .

For  $n$  even integer, we define the  $n$ -linear form of  $f_1, \dots, f_n : \mathbb{T}_\lambda \rightarrow \mathbb{C}$  associated to the multiplier  $M_n : \mathbb{R}^n \rightarrow \mathbb{C}$  by

$$\Lambda_n(M_n; f_1, \dots, f_n) := \int_{\Gamma_n(\mathbb{T}_\lambda)} M_n(k_1, k_2, \dots, k_n) \prod_{j=1}^n \widehat{f}_j(k_j)$$

and the shorthand  $\Lambda_n(M_n; f) := \Lambda_n(M_n; f, \bar{f}, \dots, f, \bar{f})$ . For example, we have

$$\begin{aligned} \int_{\mathbb{T}_\lambda} |v_x|^2 dx &= -\Lambda_2(k_1 k_2; v), \\ \operatorname{Im} \int_{\mathbb{T}_\lambda} |v|^2 v \bar{v}_x dx &= -\frac{1}{4} \Lambda_4(k_{13-24}; v). \end{aligned}$$

**Remark 2.15.** We note that

$$\overline{\Lambda(M_n; f)} = \int_{\Gamma_n(\mathbb{T}_\lambda)} \overline{M_n(k_1, k_2, \dots, k_n)} \widehat{\bar{f}}(-k_1) \widehat{f}(-k_2) \cdots \widehat{\bar{f}}(-k_{n-1}) \widehat{f}(-k_n)$$

and thus, if the multiplier  $M_n$  is such that

$$\overline{M_n}(-k_2, -k_1, \dots, -k_n, -k_{n-1}) = \sigma M_n(k_1, k_2, \dots, k_{n-1}, k_n),$$

then we have that  $\Lambda_n(M_n; f)$  is  $\mathbb{R}$ -valued ( $i\mathbb{R}$ -valued), provided that  $\sigma = +1$  ( $\sigma = -1$ ).

On the Fourier side, the equation (2.43) is written as

$$\begin{aligned} \partial_t \widehat{v}(k) = & -ik^2 \widehat{v}(k) - i \int_{k_{123}=k} k_2 \widehat{v}(k_1) \widehat{v}(k_2) \widehat{v}(k_3) - i\mu[v] \int_{k_{123}=k} \widehat{v}(k_1) \widehat{v}(k_2) \widehat{v}(k_3) \\ & + \frac{i}{2} \int_{k_{12345}=k} \widehat{v}(k_1) \widehat{v}(k_2) \widehat{v}(k_3) \widehat{v}(k_4) \widehat{v}(k_5) + i\psi[v] \widehat{v}(k). \end{aligned}$$

Also,

$$\begin{aligned} \partial_t \widehat{\bar{v}}(k) = & +ik^2 \widehat{\bar{v}}(k) - i \int_{k_{123}=k} k_2 \widehat{\bar{v}}(k_1) \widehat{\bar{v}}(k_2) \widehat{\bar{v}}(k_3) - i\mu[v] \int_{k_{123}=k} \widehat{\bar{v}}(k_1) \widehat{\bar{v}}(k_2) \widehat{\bar{v}}(k_3) \\ & - \frac{i}{2} \int_{k_{12345}=k} \widehat{\bar{v}}(k_1) \widehat{\bar{v}}(k_2) \widehat{\bar{v}}(k_3) \widehat{\bar{v}}(k_4) \widehat{\bar{v}}(k_5) - i\psi[v] \widehat{\bar{v}}(k). \end{aligned}$$

If  $n, \ell$  are even integers and  $1 \leq j \leq n$ , the elongation at index  $j$  with  $\ell$  positions of the multiplier  $M_n$  is defined by

$$\mathbb{X}_j^\ell(M_n)(k_1, k_2, \dots, k_{n+\ell}) := M_n(k_1, \dots, k_{j-1}, k_j + k_{j+1} + \dots + k_{j+\ell}, k_{j+\ell+1}, \dots, k_{n+\ell}).$$

Then, for a solution  $v$  of (2.43), we have the differentiation rule

$$\begin{aligned} \partial_t \Lambda_n(M_n; v(t)) = & i\Lambda_n \left( M_n \sum_{j=1}^n (-1)^j k_j^2; v(t) \right) \\ & - i\Lambda_{n+2} \left( \sum_{j=1}^n \mathbb{X}_j^2(M_n) k_{j+1}; v(t) \right) - i\mu[v] \Lambda_{n+2} \left( \sum_{j=1}^n \mathbb{X}_j^2(M_n); v(t) \right) \\ & + \frac{i}{2} \Lambda_{n+4} \left( \sum_{j=1}^n (-1)^{j-1} \mathbb{X}_j^4(M_n); v(t) \right). \end{aligned} \tag{2.55}$$

In comparison with the similar rule in the Euclidean setting (see [8, Proposition 3.5]), we note that the additional term (i.e. the one coupled with  $\mu[v]$ ) is due to the particularity of the gauge transformation (2.34)-(2.35). Since  $\psi$  is  $\mathbb{R}$ -valued, the terms corresponding to the  $\psi[v]v$  term in (2.43) cancel each other.

We introduce the following notation for the factor corresponding to the term  $\partial_x^2 v$  in the equation (2.43):

$$\alpha_n(\mathbf{k}) := -i(k_1^2 - k_2^2 + \dots + k_{n-1}^2 - k_n^2). \tag{2.56}$$

Note that  $\alpha_2 = 0$  on  $\Gamma_2(\mathbb{T}_\lambda)$ . A key property for the analysis of the second and third generation modified energies is the factorization of  $\alpha_4$  on  $\Gamma_4(\mathbb{T}_\lambda)$ :

$$\alpha_4(\mathbf{k}) = -i((k_1 - k_2)k_{12} + (k_3 - k_4)k_{34}) = -2ik_{12}k_{14}. \tag{2.57}$$

Furthermore, we introduce the modulations:

$$\begin{aligned} \omega_j &:= \tau_j + k_j^2, \quad \text{for } j \text{ odd}, \\ \omega_j &:= \tau_j - k_j^2, \quad \text{for } j \text{ even}, \end{aligned}$$

for all  $(\tau_1, \dots, \tau_n) \in \Gamma_n(\mathbb{R})$ , and we note that

$$\begin{aligned} \omega_1 + \omega_2 + \omega_3 + \omega_4 &= \tau_{1234} + k_1^2 - k_2^2 + k_3^2 - k_4^2 \\ &= 2k_{12}k_{14} \end{aligned}$$

which implies

$$\max_{1 \leq j \leq 4} |\omega_j| \gtrsim |k_{12}k_{14}|. \quad (2.58)$$

### 3. COERCIVITY PROPERTIES

We begin this section by revisiting the energy functional corresponding to the gauge equivalent DNLS equation (2.43) on  $\mathbb{T}_\lambda$ , namely

$$E_1[v] := \int_{\mathbb{T}_\lambda} \left( |\partial_x v|^2 - \frac{1}{2} |v|^2 \operatorname{Im}(v \partial_x \bar{v}) \right) dx + \frac{1}{2} \mu[v] \|v\|_{L^4(\mathbb{T}_\lambda)}^4 + 2\mu[v] P_1[v] - \mu[v]^2 M[v]. \quad (3.1)$$

Compared to the real line case, due to the particularity of the gauge transformation in the periodic setting, the terms coupled with the coefficient  $\mu[v]$  are new. The last two terms are integrals of motion, so we could discard them. However, the term  $\frac{1}{2} \mu[v] \|v\|_{L^4}^4$  is not conserved by the flow of (2.43).

**Remark 3.1.** If  $v$  is a smooth solution of (2.43),  $\|v\|_{L^4}$  is not necessarily conserved. Indeed, we have (see (4.8)-(4.9))

$$\begin{aligned} \partial_t \|v\|_{L^4}^4 &= 4 \operatorname{Re} \int_{\mathbb{T}_\lambda} |v|^2 \bar{v} (i \partial_x^2 v - i \mathcal{N}(v)) dx \\ &= -4 \operatorname{Im} \int_{\mathbb{T}_\lambda} |v|^2 \bar{v} \partial_x^2 v - |v|^2 \bar{v} \left( \mathcal{N}_1(v) + \mathcal{N}_2(v) + \mu[v] \mathcal{N}_3(v) - \psi[v] v \right) dx \\ &= 4 \operatorname{Im} \int_{\mathbb{T}_\lambda} \partial_x (v \bar{v}^2) \partial_x v dx - 4 \operatorname{Im} \psi[v] \|v\|_{L^4(\mathbb{T}_\lambda)}^4 + \text{h.o.t.} \\ &= 4 \operatorname{Im} \int_{\mathbb{T}_\lambda} \bar{v}^2 (\partial_x v)^2 dx + \text{h.o.t.}, \end{aligned}$$

where we used the fact that  $\psi[v]$  is  $\mathbb{R}$ -valued; see (2.44). In general, the higher order terms (h.o.t.) cannot cancel the fourth order term  $4 \operatorname{Im} \int_{\mathbb{T}_\lambda} \bar{v}^2 v_x^2 dx$ .

Nevertheless, by Sobolev embedding and interpolation of  $H^s$ -norms, we have

$$\|v\|_{L^4} \lesssim \|v\|_{H^{\frac{1}{4}}} \leq \|v\|_{L^2}^{\frac{3}{4}} \|v\|_{H^1}^{\frac{1}{4}}$$

and therefore, for any  $\varepsilon > 0$ ,

$$\frac{1}{2} \mu[v] \|v\|_{L^4}^4 \lesssim \|v\|_{L^2}^5 \|v\|_{H^1} \lesssim \varepsilon \|\partial_x v\|_{L^2}^2 + \varepsilon \|v\|_{L^2}^2 + \frac{1}{\varepsilon} \|v\|_{L^2}^{10}. \quad (3.2)$$

Therefore, we consider the essential part of the energy functional in (3.1), namely

$$\mathcal{E}[v] := \int_{\mathbb{T}_\lambda} \left( |\partial_x v|^2 - \frac{1}{2} |v|^2 \operatorname{Im}(v \bar{v}_x) \right) dx. \quad (3.3)$$

This is the same expression as the energy corresponding to (1.15) on the real line (see [7]). In view of (3.2) and the conservation of mass, when controlling the  $\dot{H}^1$ -norm of a solution  $v$  to (2.43), the above  $\mathcal{E}[v]$  is just as good as the conservation law  $E_1[v]$ .

Applying the same strategy to the mixed term  $|v|^2 \operatorname{Im}(v \bar{v}_x)$  and by using the Gagliardo-Nirenberg inequality (2.32), we get

$$\mathcal{E}[v] + 1 \gtrsim_{\delta, M[v]} (4\pi^2(1 - \varepsilon - \delta)\delta - M[v]^2) \|\partial_x v\|_{L^2}^2$$

for any  $\varepsilon, \delta > 0$ , where the constant 1 in the left hand side above hides a polynomial in  $M[v]$ . Since  $\sup_{\varepsilon, \delta > 0} (1 - \varepsilon - \delta)\delta = \frac{1}{4}$ , this would yield the mass threshold condition  $M[v] < \pi$ .

However, as was noticed by Hayashi and Ozawa [14, 15] in the Euclidean case, the choice  $\beta = \frac{3}{4}$  for the gauge transformation yields a neat expression for the corresponding energy functional and a better mass threshold condition, namely  $M[v] < 2\pi$ . In view of Remark 2.9, this mass threshold is sharp; it cannot be improved by any other gauge choice. Using the adaptation of a Gagliardo-Nirenberg inequality (Lemma 2.10 above), we show that this threshold also carries over to the periodic setting.

**Lemma 3.2.** *Let  $\lambda \geq 1$ . For any  $f \in H^1(\mathbb{T}_\lambda)$  with  $M[f] = \|f\|_{L^2(\mathbb{T}_\lambda)}^2 < 2\pi$ , we have:*

$$\|\partial_x f\|_{L^2(\mathbb{T}_\lambda)}^2 \lesssim \mathcal{E}[f] + 1. \quad (3.4)$$

*The implicit constant depends only on  $M[f]$  and blows up as  $M[f] \nearrow 2\pi$ .*

*Proof.* Consider  $g = \mathcal{G}_\beta(f)$ . Then  $|g| = |f|$  and

$$\partial_x f = e^{i\beta \mathcal{I}(g)} \left( \partial_x g + i\beta(|g|^2 - \mu[g])g \right).$$

Using (2.31), it follows that

$$\begin{aligned} \|\partial_x f\|_{L^2}^2 &= \|\partial_x g\|_{L^2}^2 + \beta^2 \|(|g|^2 - \mu[g])g\|_{L^2}^2 - 2\beta \int (|g|^2 - \mu[g]) \operatorname{Im}(g \partial_x \bar{g}) dx \\ &\leq \left( 1 + \beta^2 \|g\|_{L^2}^4 + 2|\beta| \|g\|_{L^2}^2 \right) \|\partial_x g\|_{L^2}^2. \end{aligned} \quad (3.5)$$

Straightforward computations give us

$$\begin{aligned} \mathcal{E}[f] &= \|\partial_x g\|_{L^2}^2 + \beta^2 \int (|g|^2 - \mu[g])^2 dx - 2\beta \int (|g|^2 - \mu[g]) \operatorname{Im}(g \partial_x \bar{g}) dx \\ &\quad - \frac{1}{2} \int |g|^2 \operatorname{Im}(g \partial_x \bar{g}) dx + \frac{\beta}{2} \int (|g|^2 - \mu[g]) |g|^4 dx \end{aligned}$$

By taking  $\beta = -\frac{1}{4}$ , we obtain

$$\begin{aligned} \mathcal{E}[f] &= \|\partial_x g\|_{L^2}^2 - \frac{1}{16} \|g\|_{L^6}^6 + \frac{1}{16} (\mu[g]^2 - 2\mu[g]) \|g\|_{L^2}^2 - \frac{1}{2} \mu[g] \int \operatorname{Im}(g \partial_x \bar{g}) dx + \frac{1}{8} \mu[g] \|g\|_{L^4}^4 \\ &\geq \|\partial_x g\|_{L^2}^2 - \frac{1}{16} \|g\|_{L^6}^6 - \frac{1}{8} \mu[g] \|g\|_{L^2}^2 - \frac{1}{2} \mu[g] \int \operatorname{Im}(g \partial_x \bar{g}) dx \end{aligned}$$

and note that for any  $\varepsilon > 0$  and any  $\lambda \geq 1$ ,

$$\left| \frac{1}{2} \mu[g] \int \operatorname{Im}(g \partial_x \bar{g}) dx \right| \leq \|g\|_{L^2}^3 \|\partial_x g\|_{L^2} \leq \varepsilon \|\partial_x g\|_{L^2}^2 + C_\varepsilon \|g\|_{L^2}^6,$$

for some  $C_\varepsilon \sim \varepsilon^{-1}$ . We choose  $\varepsilon > 0$  such that  $\|f\|_{L^2}^4 (\frac{1}{4\pi^2} + \frac{\varepsilon}{16}) < 1 - \varepsilon$ , and by (2.32)<sup>12</sup> we then get

$$\begin{aligned} \mathcal{E}[f] &\geq (1 - \varepsilon) \|\partial_x g\|_{L^2}^2 - \frac{1}{16} \left( \frac{4}{\pi^2} + \varepsilon \right) \|\partial_x g\|_{L^2}^2 \|g\|_{L^2}^4 - \frac{1}{16} K_\varepsilon \|g\|_{L^2}^6 \\ &\quad - \frac{1}{8} \mu[g] \|g\|_{L^2}^2 - C_\varepsilon \|g\|_{L^2}^6 \end{aligned}$$

and thus

$$\mathcal{E}[f] + M[f]^3 \gtrsim \left( (1 - \varepsilon) - \left( \frac{1}{4\pi^2} + \frac{\varepsilon}{16} \right) \|f\|_{L^2}^4 \right) \|\partial_x g\|_{L^2}^2. \quad (3.6)$$

By combining (3.5) and (3.6), we deduce (3.4) and the proof is complete.  $\square$

<sup>12</sup>. By using (2.31) at this point, the coercivity of  $\mathcal{E}[\cdot] + 1$  would be obtained under  $M[u_0] < 2\sqrt{2}$ .

Inspired by a recent paper of Guo and Wu [13], we can improve the mass threshold below which we can control the  $\dot{H}^1$ -norm of  $f$  by using both the energy  $\mathcal{E}[f]$  and the momentum

$$\mathcal{P}[f] := \int_{\mathbb{T}_\lambda} \operatorname{Im}(f \partial_x \bar{f}) dx - \frac{1}{2} \|f\|_{L^4(\mathbb{T}_\lambda)}^4 \quad (3.7)$$

associated to (2.43), where we dropped the conserved term from (2.39). The key observation is to notice that by modulating  $f$ , the change in kinetic energy incurred resembles the main part of the momentum  $\mathcal{P}[f]$  (see (3.10) below).

**Lemma 3.3.** *Let  $\lambda \geq 1$ . For any  $f \in H^1(\mathbb{T}_\lambda)$  with  $M[f] = \|f\|_{L^2(\mathbb{T}_\lambda)}^2 < 4\pi$ , we have:*

$$\|\partial_x f\|_{L^2(\mathbb{T}_\lambda)}^2 \lesssim |\mathcal{E}[f]| + \mathcal{P}[f]^2 + 1. \quad (3.8)$$

*The implicit constant depends only on  $M[f]$  and blows up as  $M[f] \nearrow 4\pi$ .*

*Proof.* As in the proof of Lemma 3.2 above, let us consider  $g = \mathcal{G}_{-\frac{1}{4}}(f)$  for which, according to (3.5), we have

$$\|\partial_x f\|_{L^2(\mathbb{T}_\lambda)} \sim \|\partial_x g\|_{L^2(\mathbb{T}_\lambda)}.$$

The main part is showing that

$$\|\partial_x g\|_{L^2(\mathbb{T}_\lambda)}^2 \lesssim |E_{\frac{3}{4}}[g]| + P_{\frac{3}{4}}[g]^2 + 1. \quad (3.9)$$

Indeed, this suffices to get (3.8) as we have

$$\begin{aligned} |P_{\frac{3}{4}}[g]| &= |P_1[f]| \leq |\mathcal{P}[f]| + \mu[f]M[f], \\ |E_{\frac{3}{4}}[g]| &= |E_1[f]| \lesssim |\mathcal{E}[f]| + \frac{1}{2}\mu[f]\|f\|_{L^4}^4 + \mu[f]|\mathcal{P}[f]| + \mu[f]^2 M[f], \end{aligned}$$

and we can use (3.2).

From (2.39)-(2.40), we recall that

$$\begin{aligned} P_{\frac{3}{4}}[g] &= \int_{\mathbb{T}_\lambda} \operatorname{Im}(g \partial_x \bar{g}) dx - \frac{1}{4} \|g\|_{L^4}^4 + \frac{3}{4} \mu[g] M[g], \\ E_{\frac{3}{4}}[g] &= \|\partial_x g\|_{L^2}^2 - \frac{1}{16} \|g\|_{L^6}^6 + \frac{3}{8} \mu[g] \|g\|_{L^4}^4 + \frac{3}{2} \mu[g] P_{\frac{3}{4}}[g] - \frac{9}{16} \mu[g]^2 M[g]. \end{aligned}$$

In order to get (3.9), we consider the modulated function  $g_\alpha(x) := e^{i\alpha x} g(x)$  with  $\alpha \in \mathbb{Z}_\lambda$  and  $\alpha > 0$  to be chosen later. We have

$$\|\partial_x g_\alpha\|_{L^2}^2 = \|\partial_x g\|_{L^2}^2 + \alpha^2 \|g\|_{L^2}^2 - 2\alpha \int_{\mathbb{T}_\lambda} \operatorname{Im}(g \partial_x \bar{g}) dx \quad (3.10)$$

and therefore

$$E_{\frac{3}{4}}[g_\alpha] - E_{\frac{3}{4}}[g] = \alpha^2 M[g] - 2\alpha \int_{\mathbb{T}_\lambda} \operatorname{Im}(g \partial_x \bar{g}) dx + \frac{3}{8} \mu[g] (P_{\frac{3}{4}}[g_\alpha] - P_{\frac{3}{4}}[g]). \quad (3.11)$$

Since

$$\operatorname{Im}(g_\alpha \partial_x \bar{g}_\alpha) = \operatorname{Im}(g \partial_x \bar{g}) - \alpha |g|^2,$$

we also have

$$P_{\frac{3}{4}}[g_\alpha] - P_{\frac{3}{4}}[g] = -\alpha M[g]. \quad (3.12)$$

Therefore

$$\frac{1}{2\alpha} E_{\frac{3}{4}}[g_\alpha] - \frac{1}{2\alpha} E_{\frac{3}{4}}[g] = \frac{\alpha}{2} M[g] - \int_{\mathbb{T}_\lambda} \operatorname{Im}(g \partial_x \bar{g}) dx - \frac{3}{4} \mu[g] M[g]$$

and thus we find that

$$\frac{1}{2\alpha}E_{\frac{3}{4}}[g_\alpha] - \frac{1}{2\alpha}E_{\frac{3}{4}}[g] - \frac{\alpha}{2}M[g] + \frac{1}{4}\|g\|_{L^4}^4 = -P_{\frac{3}{4}}[g]. \quad (3.13)$$

We now use the Gagliardo-Nirenberg inequality (2.33) to give a lower bound to the first term in (3.13); we drop the positive term  $\frac{3}{8}\mu[g_\alpha]\|g_\alpha\|_{L^4}^4$ . Also, we use (3.12), and taking into account that the Lebesgue norms of  $g_\alpha$  and  $g$  coincide, we have

$$\begin{aligned} E_{\frac{3}{4}}[g_\alpha] &\geq \|g\|_{L^6}^6 \left( C_{GN}^{-18} \left( 1 + \frac{\delta}{5\pi\lambda} \right)^{-4} \frac{\|g\|_{L^6}^{12}}{\|g\|_{L^4}^{16}} - \frac{1}{16} \right) - \frac{1}{\pi\lambda\delta}M[g] \\ &\quad + \frac{3}{2}\mu[g] \left( P_{\frac{3}{4}}[g] - \alpha M[g] \right) - \frac{9}{16}\mu[g]^2M[g] \end{aligned}$$

By (3.13), we then get

$$\begin{aligned} |P_{\frac{3}{4}}[g]| + \frac{3}{4}\mu[g]M[g] &\geq \frac{1}{4}\|g\|_{L^4}^4 - \frac{\alpha}{2}M[g] - \frac{1}{2\alpha}\varphi\left(\frac{\|g\|_{L^6}^6}{\|g\|_{L^4}^8}\right)\|g\|_{L^4}^8 \\ &\quad - \frac{1}{2\alpha} \left( |E_{\frac{3}{4}}[g]| + \frac{3}{2}\mu[g]|P_{\frac{3}{4}}[g]| + \frac{9}{16}\mu[g]^2M[g] + \frac{1}{\pi\lambda\delta}M[g] \right), \end{aligned}$$

where

$$\varphi(x) := \left( \frac{1}{16} - C_{GN}^{-18} \left( 1 + \frac{\delta}{5\pi\lambda} \right)^{-4} x^2 \right) x$$

and for which we have

$$\max_{x>0} \varphi(x) \geq \max_{x>0} \left( \frac{1}{16} - C_{GN}^{-18} x^2 \right) x = \frac{1}{64\pi}.$$

We now balance the terms  $\frac{\alpha}{2}M[g]$  and  $\frac{1}{128\pi\alpha}\|g\|_{L^4}^8$  by choosing

$$\alpha_* := \frac{\|g\|_{L^4}^4}{8\sqrt{\pi}\|g\|_{L^2}}.$$

However, in order to correctly define  $g_\alpha$  as a periodic function on  $\mathbb{T}_\lambda$ , we take

$$\alpha := \frac{1}{\lambda} \left( [\lambda\alpha_*] + 1 \right) \quad (3.14)$$

(here, by  $[x]$  we denote the integer part of  $x$ ). Then

$$\begin{aligned} -\frac{\alpha}{2}M[g] - \frac{1}{128\pi\alpha}\|g\|_{L^4}^8 &\geq -\frac{\alpha_*}{2}M[g] - \frac{1}{128\pi\alpha_*}\|g\|_{L^4}^8 - \frac{1}{2\lambda}M[g] \\ &= -\alpha_*M[g] - \frac{1}{2\lambda}M[g] \end{aligned}$$

and taking into account that  $\lambda \geq 1$ , we deduce

$$\begin{aligned} |P_{\frac{3}{4}}[g]| + M[g]^2 + M[g] &\geq \frac{1}{4}\|g\|_{L^4}^4 - \alpha_*M[g] \\ &\quad - \frac{1}{2\alpha_*} \left( |E_{\frac{3}{4}}[g]| + M[g]|P_{\frac{3}{4}}[g]| + M[g]^3 + \delta^{-1}M[g] \right). \end{aligned} \quad (3.15)$$



We consider the following positive reals

$$\begin{aligned} a &:= \frac{1}{4} \left( 1 - \frac{1}{2\sqrt{\pi}} \|g\|_{L^2} \right), \\ b &:= 4\sqrt{\pi} \|g\|_{L^2} \left( |E_{\frac{3}{4}}[g]| + M[g] |P_{\frac{3}{4}}[g]| + M[g]^3 + \delta^{-1} M[g] \right), \\ c &:= |P_{\frac{3}{4}}[g]| + M[g]^2 + M[g]. \end{aligned}$$

Thus, the inequality (3.15) provides the following

$$c \geq a \|g\|_{L^4}^4 - \frac{b}{\|g\|_{L^4}^4}.$$

It follows that

$$\|g\|_{L^4}^4 \leq \frac{c + \sqrt{c^2 + 4ab}}{2a} \lesssim c + b^{\frac{1}{2}}$$

and so we obtain

$$\|g\|_{L^4}^8 \lesssim c^2 + b \lesssim |E_{\frac{3}{4}}[g]| + |P_{\frac{3}{4}}[g]|^2 + 1. \quad (3.16)$$

Therefore, by using again (2.33),

$$\begin{aligned} \|\partial_x g\|_{L^2}^2 + \frac{2}{\delta} \mu[g] &= E_{\frac{3}{4}}[g] + \frac{1}{16} \|g\|_{L^6}^6 - \frac{3}{8} \mu[g] \|g\|_{L^4}^4 - \frac{3}{2} \mu[g] P_{\frac{3}{4}}[g] + \frac{9}{16} \mu[g]^2 M[g] + \frac{2}{\delta} \mu[g] \\ &\lesssim |E_{\frac{3}{4}}[g]| + |P_{\frac{3}{4}}[g]| + 1 + \left( \|\partial_x g\|_{L^2}^2 + \frac{2}{\delta} \mu[g] \right)^{\frac{1}{3}} \|g\|_{L^4}^{\frac{16}{3}} \end{aligned}$$

where the implicit constant can be taken to depend only on  $M[g]$ . Then either

$$\|\partial_x g\|_{L^2}^2 + \frac{2}{\delta} \mu[g] \lesssim |E_{\frac{3}{4}}[g]| + |P_{\frac{3}{4}}[g]| + 1$$

or

$$\left( \|\partial_x g\|_{L^2}^2 + \frac{2}{\delta} \mu[g] \right)^{\frac{2}{3}} \lesssim \|g\|_{L^4}^{\frac{16}{3}}$$

and we use (3.16). In both cases, (3.9) holds and the proof is completed.  $\square$

#### 4. LOCAL WELL-POSEDNESS FOR THE $I$ -SYSTEM

Given  $v$  (sufficiently smooth) solution of (2.43), since  $Iv$  does not solve the gauge equivalent equation (2.43),  $P_1[Iv]$  and  $E_1[Iv]$  are not conservation laws. Instead,  $v$  satisfies the following  $I$ -system

$$\begin{cases} i\partial_t(Iv) + \partial_x^2(Iv) = -iI(v^2\partial_x\bar{v}) - \frac{1}{2}I(|v|^4v) + \mu[v]I(|v|^2v) - \psi[v](Iv), & x \in \mathbb{T} \\ (Iv)|_{t=0} = Iv_0. \end{cases} \quad (4.1)$$

We modify the local well-posedness proof for (2.43) to obtain the following result for (4.1).

**Proposition 4.1.** *Let  $B > 0$ . There exist  $\delta \sim B^{-\theta}$  (for some  $\theta > 0$ ) and  $D > 0$  (both independent of  $N$  and  $\lambda$ ) such that if  $v_0 \in H^s(\mathbb{T}_\lambda)$  is such that  $\|Iv_0\|_{H^1(\mathbb{T}_\lambda)} \leq B$ , then*

$$\|Iv\|_{Z^1([0,\delta] \times \mathbb{T}_\lambda)} \leq D. \quad (4.2)$$

In order to prove this result, we use the estimates of the local well-posedness theory for (2.43) due to Herr [17] and an interpolation lemma of Colliander, Keel, Staffilani, Takaoka, and Tao [9, Lemma 12.1] for translation invariant multi-linear operators.

**Lemma 4.2.** [17, Section 4] *Let  $\delta \in (0, 1)$  and  $\lambda \geq 1$ . There exist  $c, \varepsilon > 0$  such that*

$$\|u_1(\partial_x \overline{u_2})u_3\|_{\tilde{Z}^{\frac{1}{2}}(\mathbb{R} \times \mathbb{T}_\lambda)} \leq c\delta^\varepsilon \prod_{j=1}^3 \|u_j\|_{X^{\frac{1}{2}, \frac{1}{2}}(\mathbb{R} \times \mathbb{T}_\lambda)}, \quad (4.3)$$

$$\|u_1 \overline{u_2} u_3 \overline{u_4} u_5\|_{\tilde{Z}^{\frac{1}{2}}(\mathbb{R} \times \mathbb{T}_\lambda)} \leq c\delta^\varepsilon \prod_{j=1}^5 \|u_j\|_{X^{\frac{1}{2}, \frac{1}{2}}(\mathbb{R} \times \mathbb{T}_\lambda)}, \quad (4.4)$$

$$\|u_1 \overline{u_2} u_3\|_{\tilde{Z}^{\frac{1}{2}}(\mathbb{R} \times \mathbb{T}_\lambda)} \leq c\delta^\varepsilon \prod_{j=1}^3 \|u_j\|_{X^{\frac{1}{2}, \frac{1}{2}}(\mathbb{R} \times \mathbb{T}_\lambda)}, \quad (4.5)$$

for all  $u_j \in \mathcal{S}_\lambda$  with  $\text{supp}(u_j) \in \{(t, x) \in \mathbb{R} \times \mathbb{T}_\lambda : |t| \leq \delta\}$ ,  $1 \leq j \leq 5$ .

**Remark 4.3.** One can check that the pointwise weights bounds provided by [17, Lemma 4.1, Lemma 4.3] hold uniformly in  $\lambda \geq 1$  (although, in view of Remark 2.1, further sub-cases have to be addressed). Then, the multi-linear estimates above use only the  $L^4$ -Strichartz and Sobolev inequalities of Lemma 2.3 above, which are all scaling invariant.

In order to state the interpolation lemma, let  $I_N^s$  denote the  $I$ -operator introduced in (2.50). Also, following [9], we let  $S_x$  to denote the shift operator  $S_x u(y, t) = u(y - x, t)$ . A Banach space  $X$  of functions  $u : J \times \mathbb{T}_\lambda \rightarrow \mathbb{C}$  (where  $J \subset \mathbb{R}$  is some time interval) is translation invariant if  $\|S_x u\|_X = \|u\|_X$  for all  $u \in X$  and all  $x$ . We use the spaces  $X = X^{1, \frac{1}{2}}(J \times \mathbb{T}_\lambda)$  and  $Z = \tilde{Z}^1(J \times \mathbb{T}_\lambda)$  which clearly satisfy this requirement. An  $n$ -linear operator  $T : X \times \dots \times X \rightarrow Z$  is translation invariant if  $S_x T(u_1, \dots, u_n) = T(S_x u_1, \dots, S_x u_n)$  for all  $u_j \in X$ .

**Lemma 4.4.** *Let  $s_0 > 0$ ,  $n \geq 1$  and let  $T : X \times \dots \times X \rightarrow Z$  be a translation invariant  $n$ -linear operator. Suppose*

$$\|I_1^s T(u_1, \dots, u_n)\|_Z \leq C \prod_{j=1}^n \|I_1^s u_j\|_X \quad (4.6)$$

for all  $s_0 \leq s \leq 1$  and all  $u_j \in X$ , for some  $C > 0$ . Then, we also have

$$\|I_N^s T(u_1, \dots, u_n)\|_Z \leq DC \prod_{j=1}^n \|I_N^s u_j\|_X \quad (4.7)$$

for all  $s_0 \leq s \leq 1$  and all  $u_j \in X$ , for some  $D > 0$  independent of  $N$  and  $\lambda$ .

To convince the reader that the proof of [9, Lemma 12.1] yields the constant  $D$  independent of the parameter  $\lambda$  (as well as  $N$ ), we provide the following remark that uses the “periodization” procedure also encountered in the Poisson summation formula.

**Remark 4.5.** We know that the Littlewood-Paley projection operators  $P_{\lesssim N} f := \phi_N * f$  are uniformly bounded in  $N$ , where  $\phi_N := N\phi(N \cdot)$  and  $\hat{\phi}$  is a symmetric function on  $\mathbb{Z}_\lambda$  equal to one on  $\{|k| \leq 1\}$  and vanishes outside  $\{|k| < 2\}$ . However, the bound  $\|\phi\|_{L^1(\mathbb{T}_\lambda)}$  depends on  $\lambda$ . We modify slightly this usual definition in order to ensure uniform boundedness in the scaling parameter  $\lambda$  as well. Thus, let  $\psi$  be a Schwarz function on  $\mathbb{R}$  such that  $\hat{\psi}$  is a symmetric bump function compactly supported in  $\{\xi \in \mathbb{R} : |\xi| \leq 2\}$  and identically one for  $|\xi| \leq 1$ . Define  $\psi_N := N\psi(N \cdot)$  and for any  $x \in \mathbb{T}_\lambda$  we set

$$\varphi_N(x) := \sum_{k \in \mathbb{Z}} \psi_N(x + 2\pi\lambda k).$$

Note that  $\widehat{\varphi_N}(k) = \widehat{\psi_N}(k)$  for any  $k \in \mathbb{Z}_\lambda$ , and thus the operator  $P_{\lesssim N}f = \varphi_N * f$  acts as the identity operator when  $\text{supp}(\widehat{f}) \subset \{k \in \mathbb{Z}_\lambda : |k| \lesssim N\}$  (this is compatible with the region where the operators  $I_N^s$  also behave like the identity operator). Also,

$$\|\varphi_N\|_{L^1(\mathbb{T}_\lambda)} = \|\psi_N\|_{L^1(\mathbb{R})} = \|\psi\|_{L^1(\mathbb{R})}$$

and therefore

$$\|P_{\lesssim N}\|_{X \rightarrow X}, \|P_{\lesssim N}\|_{Z \rightarrow Z} \lesssim 1,$$

uniformly in  $N$  and  $\lambda$ . Finally, by arguing as in [9] that  $I_1^s I_N^{2-s}$  and  $N^{s-1} I_N^s I_1^{2-s}$  are bounded (uniformly in  $N$  and  $\lambda$ ), by splitting  $u_j = P_{\lesssim N}u_j + P_{\gg N}u_j$  for each  $j$ , and by estimating each contribution separately, we obtain (4.7).

We apply the above interpolation lemma to the trilinear and quintilinear terms corresponding to the right hand side of (2.43), namely

$$\mathcal{N}(v) = -iv^2 \partial_x \bar{v} - \frac{1}{2}|v|^4 v + \mu[v]|v|^2 v - \psi[v]v \quad (4.8)$$

$$=: \mathcal{N}_1(v) + \mathcal{N}_2(v) + \mu[v]\mathcal{N}_3(v) - \psi[v]v. \quad (4.9)$$

Note that the estimates of Lemma 4.2 give (4.6) for  $s_0 = \frac{1}{2}$ . Since  $I_N^1 = \text{Id}$  for any  $N$ , we obtain the estimate (4.6) for  $s = 1$  via the Leibniz rule and Lemma 4.2. For example,

$$\begin{aligned} \|u_1(\partial_x \bar{u}_2)u_3\|_{\tilde{Z}^1} &\lesssim \|\langle \partial_x \rangle^{\frac{1}{2}} u_1(\partial_x \bar{u}_2)u_3\|_{\tilde{Z}^{\frac{1}{2}}} + \|u_1(\partial_x \langle \partial_x \rangle^{\frac{1}{2}} \bar{u}_2)u_3\|_{\tilde{Z}^{\frac{1}{2}}} + \|u_1(\partial_x \bar{u}_2)(\langle \partial_x \rangle^{\frac{1}{2}} u_3)\|_{\tilde{Z}^{\frac{1}{2}}} \\ &\lesssim \delta^\varepsilon \prod_{j=1}^3 \|\langle \partial_x \rangle^{\frac{1}{2}} u_j\|_{X^{\frac{1}{2}, \frac{1}{2}}} \sim \delta^\varepsilon \prod_{j=1}^3 \|u_j\|_{X^{1, \frac{1}{2}}}. \end{aligned}$$

One argues analogously for the other multi-linear estimates of Lemma 4.2. Hence, by applying Lemma 4.4, we obtain

$$\|I(u_1(\partial_x \bar{u}_2)u_3)\|_{\tilde{Z}^1} \lesssim \delta^\varepsilon \prod_{j=1}^3 \|Iu_j\|_{X^{1, \frac{1}{2}}}, \quad (4.10)$$

$$\|I(u_1 \bar{u}_2 u_3 \bar{u}_4 u_5)\|_{\tilde{Z}^1} \lesssim \delta^\varepsilon \prod_{j=1}^5 \|Iu_j\|_{X^{1, \frac{1}{2}}}, \quad (4.11)$$

$$\|I(u_1 \bar{u}_2 u_3)\|_{\tilde{Z}^1} \lesssim \delta^\varepsilon \prod_{j=1}^3 \|Iu_j\|_{X^{1, \frac{1}{2}}}. \quad (4.12)$$

We also need the following Lipschitz continuity properties for the coupling coefficients  $\mu[v]$  and  $\psi[v]$ . We easily have

$$|\mu[f] - \mu[g]| \leq \frac{1}{2\pi\lambda} (\|f\|_{L^2(\mathbb{T}_\lambda)} + \|g\|_{L^2(\mathbb{T}_\lambda)}) \|f - g\|_{L^2(\mathbb{T}_\lambda)}, \quad (4.13)$$

while Hölder's inequality, Parseval's identity, and the  $L^6$ -Sobolev inequality give

$$|\psi[f] - \psi[g]| \lesssim \frac{1}{2\pi\lambda} \left( \|f\|_{H^{\frac{1}{2}}}^3 + \|f\|_{L^2} + \|g\|_{H^{\frac{1}{2}}}^3 + \|g\|_{L^2} \right) \|f - g\|_{H^{\frac{1}{2}}(\mathbb{T}_\lambda)}. \quad (4.14)$$

The reader can also consult [17, Lemma 2.5].

We can now proceed with the proof of Proposition 4.1 by using the fixed point argument in a closed ball of the space  $W = \{v : \eta_\delta(t)Iv(t, x) \in Z^1(\mathbb{R} \times \mathbb{T}_\lambda)\}$  with norm

$$\|v\|_W := \|\eta_\delta Iv\|_{Z^1(\mathbb{R} \times \mathbb{T}_\lambda)},$$

with  $\delta \in (0, 1)$  and  $D > 0$  to be chosen later, and  $\eta_\delta(t) := \eta(\frac{t}{\delta})$ . Grouping terms as in (2.45)-(2.46), and using the Duhamel formulation, solutions of (4.1) are those  $v$  that satisfy

$$Iv(t) = U_\lambda(t)Iv_0 - i \int_0^t U_\lambda(t-t')IN(v(t'))dt' \quad (4.15)$$

in the  $C([0, T]; H^1(\mathbb{T}_\lambda))$  topology, for some  $T > 0$ . Consider the mapping  $v \mapsto \Gamma(v)$  given by

$$\Gamma(v) := \eta(t)U_\lambda(t)v_0 - i\eta(t) \int_0^t U_\lambda(t-t')\mathcal{N}(\eta_\delta(t')v(t'))dt'.$$

By (2.12)-(2.13) and (4.10)-(4.12), we have

$$\begin{aligned} \|\Gamma(v)\|_W &\leq \|\eta(t)U_\lambda(t)Iv_0\|_{Z^1} + \left\| \eta(t) \int_0^t U_\lambda(t-\tau)IN(\eta_\delta(t')v(t'))d\tau \right\|_{Z^1} \\ &\leq c_1 (\|Iv_0\|_{H^1(\mathbb{T}_\lambda)} + \|\mathcal{N}(\eta_\delta v)\|_{\tilde{Z}^1}) \\ &\leq c_1 B + c_2 \delta^\varepsilon \left( \|\eta_\delta Iv\|_{X^{1, \frac{1}{2}}}^3 + \|\eta_\delta Iv\|_{X^{1, \frac{1}{2}}}^5 + \|\eta_\delta Iv\|_{\tilde{Z}^1} \right). \end{aligned} \quad (4.16)$$

Also,

$$\begin{aligned} \|\Gamma(v_1) - \Gamma(v_2)\|_W &= \left\| \eta(t) \int_0^t U_\lambda(t-\tau) (IN(\eta_\delta(t')v_1(t')) - IN(\eta_\delta(t')v_2(t'))) d\tau \right\|_{Z^1} \\ &\lesssim \|I(\mathcal{N}_1(\eta_\delta v_1) - \mathcal{N}_1(\eta_\delta v_2))\|_{\tilde{Z}^1} + \|I(\mathcal{N}_2(\eta_\delta v_1) - \mathcal{N}_2(\eta_\delta v_2))\|_{\tilde{Z}^1} \\ &\quad + \|I(\mathcal{N}_3(\eta_\delta v_1) - \mathcal{N}_3(\eta_\delta v_2))\|_{\tilde{Z}^1} + \|I(\eta_\delta v_1 - \eta_\delta v_2)\|_{\tilde{Z}^1}. \end{aligned}$$

We write

$$\mathcal{N}_1(u_1) - \mathcal{N}_1(u_2) = u_1(\partial_x \overline{u_1})(u_1 - u_2) + u_1 \partial_x (\overline{u_1 - u_2})u_2 + (u_1 - u_2)(\partial_x \overline{u_2})u_2$$

and by using (4.10), we obtain

$$\|I(\mathcal{N}_1(\eta_\delta v_1) - \mathcal{N}_1(\eta_\delta v_2))\|_{\tilde{Z}^1} \lesssim \delta^\varepsilon (\|\eta_\delta Iv_1\|_{Z^1}^2 + \|\eta_\delta Iv_2\|_{Z^1}^2) \|\eta_\delta I(v_1 - v_2)\|_{Z^1}.$$

By using (4.11) and (4.12), we similarly have

$$\begin{aligned} \|I(\mathcal{N}_2(\eta_\delta v_1) - \mathcal{N}_2(\eta_\delta v_2))\|_{\tilde{Z}^1} &\lesssim \delta^\varepsilon (\|\eta_\delta Iv_1\|_{Z^1}^4 + \|\eta_\delta Iv_2\|_{Z^1}^4) \|\eta_\delta I(v_1 - v_2)\|_{Z^1}, \\ \|I(\mathcal{N}_3(\eta_\delta v_1) - \mathcal{N}_3(\eta_\delta v_2))\|_{\tilde{Z}^1} &\lesssim \delta^\varepsilon (\|\eta_\delta Iv_1\|_{Z^1}^2 + \|\eta_\delta Iv_2\|_{Z^1}^2) \|\eta_\delta I(v_1 - v_2)\|_{Z^1}. \end{aligned}$$

It follows that

$$\|\Gamma(v_1) - \Gamma(v_2)\|_W \lesssim \delta^\varepsilon (\|v_1\|_W^2 + \|v_2\|_W^2 + \|v_1\|_W^4 + \|v_2\|_W^4) \|v_1 - v_2\|_W. \quad (4.17)$$

By taking  $D = 2c_1 B + 1$  and  $\delta$  such that  $\delta^\varepsilon D^5 \sim 1$ , from (4.16) and (4.17), we get

$$\|\Gamma(v)\|_W \leq D \quad \text{and} \quad \|\Gamma(v_1) - \Gamma(v_2)\|_W \leq \frac{1}{2} \|v_1 - v_2\|_W,$$

for all  $v, v_1, v_2 \in \{w \in W : \|w\|_W \leq D\}$ . Hence, by Banach's fixed point theorem, there exists a unique  $v$  with  $\|v\|_W \leq D$  such that  $v = \Gamma(v)$  in  $W$ . Thus,

$$\|Iv\|_{Z^1([0, \delta] \times \mathbb{T}_\lambda)} \leq \|\eta_\delta Iv\|_{Z^1(\mathbb{R} \times \mathbb{T}_\lambda)} \leq D$$

and it follows that (4.15) holds on  $[0, \delta]$ . The proof of Proposition 4.1 is completed.

## 5. MODIFIED ENERGY FUNCTIONALS VIA THE $I$ -OPERATOR AND CORRECTION TERMS

In view of the discussion in Section 3, we consider the essential part of the energy functional associated to (2.43), namely

$$\mathcal{E}[v] := \int_{\mathbb{T}_\lambda} \left( |\partial_x v|^2 - \frac{1}{2} |v|^2 \operatorname{Im}(v \bar{v}_x) \right) dx. \quad (5.1)$$

The *first modified energy* is defined to be the  $\mathbb{R}$ -valued functional

$$\mathcal{E}^1[v] := \mathcal{E}[Iv] = -\Lambda_2(k_1 k_2 m_1 m_2; v) + \frac{1}{4} \Lambda_4(k_{13} m_1 m_2 m_3 m_4; v) \quad (5.2)$$

and for  $v$  sufficiently smooth solution of (2.43), one can compute its time increment from the fundamental theorem of calculus

$$\mathcal{E}^1[v(t_0 + \delta)] - \mathcal{E}^1[v(t_0)] = \int_{t_0}^{t_0 + \delta} \frac{d}{dt} \mathcal{E}^1[v(t)] dt. \quad (5.3)$$

Using (2.55), we have

$$\frac{d}{dt} \mathcal{E}^1[v(t)] = \Lambda_4(M_4^1; v) + \Lambda_6(M_6^1; v) + \Lambda_8(M_8^1; v) - i\mu[v] \left( \Lambda_4(K_4^1; v) + \Lambda_6(K_6^1; v) \right), \quad (5.4)$$

with the multipliers  $M_4^1, M_6^1, M_8^1$  given by [7, Proposition 4.1], e.g.

$$M_4^1(\mathbf{k}) := -\frac{i}{2} m_1 m_2 m_3 m_4 k_{12} k_{13} k_{14} - \frac{i}{2} (m_1^2 k_1^2 k_3 + m_2^2 k_2^2 k_4 + m_3^2 k_3^2 k_1 + m_4^2 k_4^2 k_2). \quad (5.5)$$

Here, it is not particularly important to have the precise expression of the multipliers  $M_6^1, M_8^1$ . The multipliers  $K_4^1, K_6^1$  are new to the periodic setting (due to a different expression of the gauge transformation) and are given by

$$K_4^1(\mathbf{k}) := \frac{1}{2} \sum_{j=1}^4 (-1)^j m_j^2 k_j^2, \quad (5.6)$$

$$K_6^1(\mathbf{k}) := \frac{2}{3} \sum_{\substack{\{a,c,e\}=\{1,3,5\} \\ \{b,d,f\}=\{2,4,6\}}} m_a m_b m_c m_{def} - m_d m_e m_f m_{abc}. \quad (5.7)$$

Note that by Remark 2.15,  $\Lambda_4(K_4^1; v)$  and  $\Lambda_6(K_6^1; v)$  are purely imaginary, and that  $\Lambda_4(M_4^1; v)$ ,  $\Lambda_6(M_6^1; v)$  and  $\Lambda_8(M_8^1; v)$  are  $\mathbb{R}$ -valued.

The rule of thumb when one tries to prove estimates on the various terms of (5.3) is that “different pieces of  $\Lambda_n$  appearing in the right hand side of  $\partial_t \mathcal{E}^1(v)$  are easier for  $n$  larger” [8, p. 72]. This motivates the following procedure when one tries to refine the  $I$ -method.

A *second instantiation of the  $I$ -method* modifies further the expression of the energy functional by taking

$$\mathcal{E}^2[v] := \mathcal{E}^1[v] + \Lambda_4(\sigma_4; v) \quad (5.8)$$

where the “correction” multiplier  $\sigma_4$  is chosen so that in the expression of  $\frac{d}{dt} \mathcal{E}^2(v)$ , no fourth order term  $\Lambda_4(\cdot; v)$  appears. For the sake of keeping the equations compact, we choose to drop the reference to  $v$  from  $\Lambda_n(M_n; v)$ , and the frequency arguments  $\mathbf{k} = (k_1, \dots, k_n)$  when the formulae get too long.

By the differentiation rule (2.55), we have

$$\begin{aligned} \frac{d}{dt} \Lambda_4(\sigma_4) &= \Lambda_4(\sigma_4 \alpha_4) - i\Lambda_6 \left( \sum_{j=1}^4 \mathbb{X}_j^2(\sigma_4) k_{j+1} \right) + \frac{i}{2} \Lambda_8 \left( \sum_{j=1}^4 (-1)^{j-1} \mathbb{X}_j^4(\sigma_4) \right) \\ &\quad - i\mu[v] \Lambda_6 \left( \sum_{j=1}^4 \mathbb{X}_j^2(\sigma_4) \right). \end{aligned} \quad (5.9)$$

Note that if  $\alpha_4(\mathbf{k}) = 0$ , then either  $k_{12} = 0$  or  $k_{14} = 0$ , and both imply that  $\widetilde{M}_4(\mathbf{k}) = 0$ . We define the first correction  $\sigma_4 = \sigma_4(\mathbf{k})$  for  $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$  by setting

$$\sigma_4 := -\frac{M_4^1}{\alpha_4} = -\frac{1}{4} \left( m_1 m_2 m_3 m_4 k_{13} + \frac{m_1^2 k_1^2 k_3 + m_2^2 k_2^2 k_4 + m_3^2 k_3^2 k_1 + m_4^2 k_4^2 k_2}{k_{12} k_{14}} \right) \quad (5.10)$$

when  $\alpha_4 \neq 0$ , and  $\sigma_4 = 0$  when  $\alpha_4 = 0$ . Thus, in the second iteration of the  $I$ -method there are no resonances for the correction term as we have  $|M_4^1(\mathbf{k})| \lesssim |\alpha_4(\mathbf{k})|$  for all  $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$ .

Therefore, by (5.2), (5.8) and (5.10), the second generation modified energy is given by

$$\mathcal{E}^2[v] = -\Lambda_2(k_1 k_2 m_1 m_2) + \frac{1}{2} \Lambda_4(M_4), \quad (5.11)$$

where we set

$$M_4 := -\frac{m_1^2 k_1^2 k_3 + m_2^2 k_2^2 k_4 + m_3^2 k_3^2 k_1 + m_4^2 k_4^2 k_2}{2k_{12}k_{14}} \quad (5.12)$$

when the denominator does not vanish. Note that since  $k_{12}k_{14} = 0$  in  $\Gamma_4(\mathbb{T}_\lambda)$  implies  $m_1^2 k_1^2 k_3 + m_2^2 k_2^2 k_4 + m_3^2 k_3^2 k_1 + m_4^2 k_4^2 k_2 = 0$ , we can set in this cases  $M_4 := 0$ .

Hence from (5.4) and (5.9), we get

$$\frac{d}{dt} \mathcal{E}^2[v(t)] = \Lambda_6(M_6^2) + \Lambda_8(M_8^2) - i\mu[v] \left( \Lambda_4(K_4^1) + \Lambda_6(K_6^1) + \Lambda_6(K_6^2) \right), \quad (5.13)$$

where  $M_6^2$  and  $M_8^2$  are the multipliers given by

$$M_6^2 := \frac{i}{6} \sum_{j=1}^6 (-1)^j m_j^2 k_j^2 \quad (5.14)$$

$$- \frac{i}{72} \sum_{\substack{\{a,c,e\}=\{1,3,5\} \\ \{b,d,f\}=\{2,4,6\}}} \left( M_4(k_{abc}, k_d, k_e, k_f) k_b + M_4(k_a, k_{bcd}, k_e, k_f) k_c \right. \\ \left. + M_4(k_a, k_b, k_{cde}, k_f) k_d + M_4(k_a, k_b, k_c, k_{def}) k_e \right),$$

$$M_8^2 := C_8 \sum_{\substack{\{a,c,e,g\}=\{1,3,5,7\} \\ \{b,d,f,h\}=\{2,4,6,8\}}} \left( M_4(k_{abcde}, k_f, k_g, k_h) - M_4(k_a, k_{bcdef}, k_g, k_h) \right. \\ \left. + M_4(k_a, k_b, k_{cdefg}, k_h) - M_4(k_a, k_b, k_c, k_{defgh}) \right) \quad (5.15)$$

(as in [8, Proposition 3.7] or [31, p. 2173]). Also,

$$K_6^2 := \sum_{j=1}^4 \mathbb{X}_j^2(\sigma_4). \quad (5.16)$$

We note that when proving the estimates on  $M_6$  (see Lemma 5.4), cancelations between the large terms coming from the first term in (5.14) and the large terms coming from the sum of  $M_4$ 's are exploited, and thus the coefficients of the two pieces of  $M_6$  are critical, whereas the constant  $C_8 = -\frac{i}{2(5!)^2}$  is irrelevant in our analysis.

**Remark 5.1 (Small Frequencies Remark).** Notice that if  $|k_j| \ll N$  for all  $j$ , we have  $m(k_j) = 1$  and thus

$$M_4(\mathbf{k}) = -\frac{k_1 k_3 k_{13} + k_2 k_4 (-k_{13})}{2k_{12}k_{14}} = \frac{k_{13}}{2}, \text{ for all } \mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda). \quad (5.17)$$

One can similarly check that if  $|k_j| \ll N$  for all  $j$ , all the multipliers  $M_n^g$ ,  $K_n^g$  ( $n = 4, 6, 8$ ,  $g = 1, 2$ ) vanish.

On  $\Gamma_n(\mathbb{T}_\lambda)$ , the largest two frequencies must have comparable sizes and thus, without loss of generality, we may assume that

$$\mathbf{k} \in \Upsilon_n(\mathbb{T}_\lambda) := \{(k_1, \dots, k_n) \in \mathbb{Z}_\lambda^n : |k_1^*| \sim |k_2^*| \gtrsim N\}, \quad (5.18)$$

where  $N$  is the frequency size threshold of the  $I$ -operator as defined in Subsection 2.4, and  $(k_1^*, \dots, k_n^*)$  denotes a rearrangement of  $(k_1, \dots, k_n)$  such that

$$|k_1^*| \geq |k_2^*| \geq \dots \geq |k_n^*|.$$

We'll also adopt the notation  $N_j = |k_j^*|$ .

Due to Remark 5.1, when proving the necessary estimates, it is enough to consider  $\mathbf{k} \in \Upsilon_n(\mathbb{T}_\lambda)$ , i.e. only the region  $N_1 \sim N_2 \gtrsim N$ .

**Remark 5.2 (Symmetry Remark).** We point out that the multipliers  $M_n^g$ 's that appear throughout this article, and consequently the associated multilinear forms  $\Lambda_n(M_n^g; v_1, v_2, \dots, v_n)$  are invariant under permutations of the even or of the odd  $k_j$  (or  $v_j$ ) indices. Also, the same is true (up to sign) if one swaps the set of all odd  $k_j$ 's (or  $v_j$ 's) with the set of all even  $k_j$ 's (respectively  $v_j$ 's).

Hence, in addition to (5.18), without loss of generality we may assume that

$$|k_1| \geq |k_3| \geq \dots \geq |k_{n-1}|, \quad |k_2| \geq |k_4| \geq \dots \geq |k_n|$$

and

$$|k_1| \geq |k_2|.$$

If all these are in place, we have  $k_1^* = k_1$ , but either  $k_2^* = k_2$  or  $k_2^* = k_3$ .

**5.1. Pointwise bounds on the multipliers.** We provide here the multiplier estimates that are relevant in our analysis, namely for the almost conservation estimates of the modified energy functional in Section 6 and in the estimates of the correction terms in Section 7. We recall that we work under the symmetry assumptions on the multipliers  $M_n^g$ ,  $K_n^g$  mentioned in Remark 5.2. Also, since we rely on (2.51), the assumption  $s \geq \frac{1}{2}$  is needed for all of the results below.

Although the multiplier  $M_4$  is not involved directly in (5.13), the refined bounds (ii) and (iii) below are crucial for  $M_6^2$  and  $M_8^2$ .

**Lemma 5.3.** [8, Lemma 4.1, 4.2] *For  $M_4$  defined by (5.12) and  $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$ , we have:*

- (i)  $|M_4(\mathbf{k})| \lesssim m(N_1)^2 N_1$ ;
- (ii) if  $|k_1| \sim |k_3| \gtrsim N \gg N_3$ , then  $|M_4(\mathbf{k})| \lesssim m(N_1)^2 N_3$ ;
- (iii) if  $|k_1| \sim |k_2| \gtrsim N \gg N_3$ , then  $M_4(\mathbf{k}) = \frac{m(k_1)^2 k_2^2}{2k_1} + O(N_3)$ .

By using the estimate (i) above, one can immediately obtain a crude bound for the symbol  $M_6^2$  (see (i) below). We recall that in [8], the refined estimate (ii) below, as well as using Bourgain's trick to provide additional denominators, make possible the global well-posedness result of (1.1) on the real line for  $s > \frac{1}{2}$ , but not at the end-point  $s = \frac{1}{2}$ . It is worth mentioning that for (ii), in the case  $N_3 \ll N$  and the largest two frequencies have same parity, it was exploited the cancellation "between the large terms coming from  $\beta_6$  and the large terms of the sum of the  $M_4$ ." Hence the almost conservation estimate of  $\mathcal{E}^2$  owes to the specific nonlinear structure  $-iv^2 \partial_x \bar{v} - \frac{1}{2}|v|^4 v$  of the gauged DNLS equation (2.43) in the Euclidean case.

**Lemma 5.4.** [8, Lemma 6.2] *For  $M_6^2$  defined by (5.14) and  $\mathbf{k} \in \Gamma_6(\mathbb{T}_\lambda)$ , we have:*

- (i)  $|M_6^2(\mathbf{k})| \lesssim m(N_1)^2 N_1^2$ ;
- (ii) if  $N_3 \ll N$ , then  $|M_6^2(\mathbf{k})| \lesssim N_1 N_3$ .

**Lemma 5.5.** *For  $\sigma_4$  defined by (5.10) and  $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$ , we have:*

$$|\sigma_4(\mathbf{k})| \lesssim m(N_1)^2 N_1.$$

*Proof.* For  $\sigma_4$ , one easily notes that  $\sigma_4^1 := -\frac{1}{4}m_1m_2m_3m_4k_{13}$  is bounded by  $m(N_1)^2N_1$  and for  $\sigma_4^2 := \sigma_4 - \sigma_4^1$ , we have Lemma 5.3 which gives  $|\sigma_4^2| \sim |M_4| \lesssim m(N_1)^2N_1$ .  $\square$

Another immediate consequence of Lemma 5.3 is the following:

**Lemma 5.6.** *For  $M_8^2$  defined by (5.15) and  $\mathbf{k} \in \Gamma_8$ , we have:*

- (i)  $|M_8^2(\mathbf{k})| \lesssim m(N_1)^2N_1$ ;
- (ii) if  $N_3 \ll N$ , then  $|M_8^2(\mathbf{k})| \lesssim N_3$ .

**Lemma 5.7.** *For  $K_4^1$  defined by (5.6) and  $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$ , we have*

- (i)  $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2N_1^2$ ;
- (ii) if  $|k_1| \sim |k_2| \gtrsim N \gg N_3$ , then  $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2N_1N_3$ .

*Proof.* The first statement is immediate as  $\xi \mapsto m(\xi)^2\xi^2$  is increasing. For the second statement,  $|m'(\xi)| \sim m(\xi)|\xi|^{-1}$  when  $|\xi| \gg N$ , and by the mean value theorem

$$|m(k_1)^2k_1^2 - m(k_2)^2k_2^2| \sim m^2(\theta)|\theta||k_1 - (-k_2)|$$

for some  $\theta$  between  $k_1$  and  $-k_2$ ; hence  $|\theta| \sim N_1$  and  $m(\theta)^2 \sim m(N_1)^2$ . Since we also have  $|k_{12}| = |k_{34}| \lesssim N_3$ , we get  $|m(k_1)^2k_1^2 - m(k_2)^2k_2^2| \lesssim m(N_1)^2N_1N_3$ . Then, the crude bound

$$|m(k_3)^2k_3^2 - m(k_4)^2k_4^2| \leq m(k_3)^2k_3^2 + m(k_4)^2k_4^2 \lesssim m(N_3)^2N_3^2$$

together with  $m(N_3)^2N_3 \leq m(N_1)^2N_1$ , concludes the proof.  $\square$

For the last lemma in this section, the first statement is immediate from  $0 < m(\cdot) \leq 1$ , while the second follows from Lemma 5.5.

**Lemma 5.8.** *For  $K_6^1, K_6^2$  defined by (5.7), (5.16) respectively, and  $\mathbf{k} \in \Gamma_6(\mathbb{T}_\lambda)$ , we have*

- (i)  $|K_6^1(\mathbf{k})| \lesssim 1$ ;
- (ii)  $|K_6^2(\mathbf{k})| \lesssim m(N_1)^2N_1$ .

**5.2. Necessity of the third iteration of the  $I$ -method.** To make the matters clear why we need to implement a third generation  $I$ -method, we prove here the decay estimate for  $\int \Lambda_6(M_6^2)dt$ . This part serves two purposes: first, to see how one applies the bilinear estimate in order to recover the result of [49, Lemma 7.5], and second to uncover the worst case scenarios and hence motivate the non-resonant subregions of Subsection 5.4.

**Proposition 5.9.** *For  $s > \frac{1}{2}$  and  $M_6^2$  defined by (5.14), we have the estimate*

$$\left| \int_0^\delta \Lambda_6(M_6^2; v(t)) dt \right| \lesssim N^{-1+\lambda^{-1+}} \|Iv\|_{Z^1([0,\delta] \times \mathbb{T}_\lambda)}^6. \quad (5.19)$$

*Proof.* We write  $v = \sum_{k \in \mathbb{Z}_\lambda} v_j$ , with  $\text{supp}(\widehat{v}_j) \subset \{(\tau, k) \in \mathbb{R} \times \mathbb{Z}_\lambda : |k| \sim N_j\}$  for each  $N_j$  dyadic number. Thus, it is enough to estimate

$$\int_{\mathbb{R}} \Lambda_6(M_6^2; v_1, v_2, \dots, v_6) dt \quad (5.20)$$

where without loss of generality we can assume, in addition to the frequency localization, that each  $\widehat{v}_j$  is real valued and non-negative. This step, as well as why it is enough to consider the time integral on  $\mathbb{R}$  rather than on  $[0, \delta]$  can be justified by standard arguments as in Section 6.



**Case 1.**  $N_1 \sim N_2 \gtrsim N \gg N_3$ . By Lemma 5.4 (ii) we have  $|M_6^2| \lesssim N_1 N_3$ . Notice that for  $s > \frac{1}{2}$  and  $\varepsilon > 0$  small enough, one obtains

$$m(N_1)^2 N_1 = N^{2-2s} N_1^{2s-1} = N^{1-\varepsilon} \left( \frac{N_1}{N} \right)^{2s-1-\varepsilon} N_1^\varepsilon \gtrsim N^{1-\varepsilon} N_1^\varepsilon. \quad (5.21)$$

In this case,  $m(N_j) = 1$  for  $j \geq 3$  and therefore by (2.25) and (2.10), we get

$$\begin{aligned} (5.20) &\lesssim \int_* \int_{**} \frac{1}{m(N_1)^2 N_1 \prod_{j=4}^6 \langle k_j \rangle} \prod_{j=1}^6 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-1+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x I v_3)(J_x I v_2)(J_x I v_4)(I v_5)(I v_6) dx dt \\ &\lesssim \frac{N^{-1+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L_{t,x}^2} \|(J_x I v_2)(J_x I v_4)\|_{L_{t,x}^2} \|I v_5\|_{L_{t,x}^\infty} \|I v_6\|_{L_{t,x}^\infty} \\ &\lesssim \frac{N^{-1+\lambda^{-1+}}}{N_1^{0+}} \prod_{j=1}^4 \|I v_j\|_{X^{1,\frac{1}{2}}} \prod_{j=5,6} \|I v_j\|_{Y^{\frac{1}{2}+,0}} \\ &\lesssim \frac{N^{-1+\lambda^{-1+}}}{N_1^{0+}} \prod_{j=1}^6 \|I v_j\|_{Z^1} \end{aligned} \quad (5.22)$$

where  $\int_*$  and  $\int_{**}$  stand for integration on  $\Gamma_6(\mathbb{R})$  and on  $\Gamma_6(\mathbb{T}_\lambda)$ , respectively (see Section 2.5). The operator  $J_x$  denotes the Bessel potential operator, i.e.  $\widehat{J_x f}(k) = \langle k \rangle \widehat{f}(k)$ .

**Remark 5.10.** For  $s = \frac{1}{2}$ , we only have  $m(N_1)^2 N_1 \gtrsim N$  as we cannot afford to borrow an  $N_1^\varepsilon$  factor as in (5.21) above. Notice that since there are no other tools to obtain additional decaying factors, to make up for the logarithmic loss in  $\lambda$ , as well as to ensure summability, one would need to obtain a better estimate, for example

$$|M_6^2| \lesssim N_1^{1-\theta} N_3^{1+\theta}, \quad (5.23)$$

which gives the following factor in (5.22):

$$N^{-1} \lambda^{-1} \frac{N_3^\theta \lambda^{0+}}{N_1^\theta} \lesssim \frac{N^{-1} \lambda^{-1-}}{N_3^{0+}}$$

(recall that since  $s \geq \frac{1}{2}$ , we have  $1 \leq \lambda \leq N$ ). We note that the decaying factor  $N^{-1} \lambda^{-1-}$  would allow us to obtain the global well-posedness result at  $s = \frac{1}{2}$  (see Section 8). Although the bound (5.23) is not conceivable on the entire  $\Gamma_6(\mathbb{T}_\lambda)$ , such an estimate can be established on a carefully chosen subset (see Section 5.4).

**Case 2.**  $N_3 \gtrsim N \gg N_4$ . By Lemma 5.4 (i) we have  $|M_6^2| \lesssim m(N_1)^2 N_1^2$ , and for  $s \geq 0$ ,  $m(N_3) N_3 \gtrsim N^{-1+} N_3^{0+}$ . We then have

$$\begin{aligned} (5.20) &\lesssim \int_* \int_{**} \frac{1}{m(N_3) N_3 \prod_{j=4}^6 \langle k_j \rangle} \prod_{j=1}^6 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-1+}}{N_3^{0+}} \int_* \int_{**} \widehat{J_x I v_1} \widehat{J_x I v_2} \widehat{J_x I v_3} \widehat{I v_4} \widehat{I v_5} \widehat{I v_6}. \end{aligned}$$

At this point we have to discuss the frequency separation of the first three factors.

**Subcase 2.1** If  $N_3 \sim N_1$ , then since  $N_3 \gg N_4$ , two out of the three frequencies  $k_1, k_2, k_3$  must have opposite signs, say  $k_1$  and  $k_2$ . Thus  $J_x I v_1$  and  $J_x I v_2$  are separated in frequency, and so are  $J_x I v_3$  and  $J_x I v_4$ . We have

$$\begin{aligned}
 (5.20) &\lesssim \frac{N^{-1+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x I v_2)(J_x I v_3)(J_x I v_4)(I v_5)(I v_6) dx dt \\
 &\lesssim \frac{N^{-1+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_2)\|_{L_{t,x}^2} \|(J_x I v_3)(J_x I v_4)\|_{L_{t,x}^2} \|I v_5\|_{L_{t,x}^\infty} \|I v_6\|_{L_{t,x}^\infty} \\
 &\lesssim \frac{N^{-1+} \lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^6 \|I v_j\|_{Z^1}.
 \end{aligned}$$

**Subcase 2.2** If  $N_3 \ll N_1$ , then as in Case 1, we can clearly apply the bilinear estimate (2.25) to the  $L_{t,x}^2$ -norms of both  $(J_x I v_1)(J_x I v_3)$  and  $(J_x I v_2)(J_x I v_4)$  and obtain

$$(5.20) \lesssim \frac{N^{-1+} \lambda^{-1+}}{N_3^{0+}} \|I v_1\|_{X^{1,\frac{1}{2}}} \|I v_2\|_{X^{1,\frac{1}{2}}} \prod_{j=3}^6 \|I v_j\|_{Z^1}.$$

Notice that in this sub case the factor  $1/N_3^{0+}$  does not allow direct summation over the dyadic numbers  $N_1 \sim N_2$ . However, exploiting the  $L^2$ -based norm of the space  $X^{1,\frac{1}{2}}$  of the first two factors, one can recover the claim (see Section 6) without any setback.

**Remark 5.11.** Notice that although in Case 2 we have three large frequencies ( $N_3 \gtrsim N \gg N_4$ ), the bound on the weight  $M_6^2$  is worse than in Case 1, and overall we obtain the same (insufficient) decaying factor of  $N^{-1+} \lambda^{-1+}$ . Therefore we also need to correct for this case.

**Case 3.**  $N_4 \gtrsim N$ . By Lemma 5.4 (i) we have  $|M_6^2| \lesssim m(N_1)^2 N_1^2$ , and for  $s \geq 0$ ,  $m(N_j) N_j \gtrsim N^{-1+} N_j^{0+}$ ,  $j = 3, 4$ . It follows that

$$\begin{aligned}
 (5.20) &\lesssim \int_{**} \int_{**} \frac{1}{m(N_3) N_3 m(N_4) N_4 \prod_{j=5,6} \langle k_j \rangle} \prod_{j=1}^6 \widehat{J_x I v_j} \\
 &\lesssim \frac{N^{-2+}}{N_3^{0+}} \int_{**} \int_{**} \widehat{J_x I v_1} \widehat{J_x I v_2} \widehat{J_x I v_3} \widehat{J_x I v_4} \widehat{I v_5} \widehat{I v_6}.
 \end{aligned}$$

Although when  $\lambda \sim N$ , the decaying factor obtained above is just as good as that in the previous cases, we can gain here another decaying factor  $\lambda^{-\frac{1}{2}+}$  by separating the analysis into subcases  $N_3 \sim N_1$  and  $N_3 \ll N_1$ , as we did in Case 2. We obtain

$$(5.20) \lesssim \frac{N^{-2+} \lambda^{-\frac{1}{2}+}}{N_3^{0+}} \|I v_1\|_{X^{1,\frac{1}{2}}} \|I v_2\|_{X^{1,\frac{1}{2}}} \prod_{j=3}^6 \|I v_j\|_{Z^1}$$

and since we choose the parameters so that  $1 \leq \lambda \leq N$ , we have in this case a better decaying factor.  $\square$

The other sixth order term in (5.13) is  $\mu[v] \Lambda_4(K_4^1; v)$ . The coefficient  $\mu[v] = \frac{1}{2\pi\lambda} \|v\|_{L^2(\mathbb{T}_\lambda)}^2$  already provides a decaying factor of  $\lambda^{-1}$ . In the remark below, we investigate the worst case scenario corresponding to this term.

**Remark 5.12.** The pointwise bound  $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2 N_1^2$  is optimal in the case  $N_3 \ll N$  and the largest two frequencies have the same parity (as we have, for example,  $|K_4^1(N_1, 0, -N_1, 0)| = m(N_1)^2 N_1^2$ ). In this case, the best estimate that can be obtained

is

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_4(K_4^1; v_1, \dots, v_4) dt &\lesssim \int_* \int_{**} \frac{1}{\langle k_3 \rangle \langle k_4 \rangle} \prod_{j=1}^4 \widehat{J_x I v_j} \\
&\lesssim \frac{1}{N_3^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_x I v_1) (J_x^{0+} I v_3) (J_x I v_2) (I v_4) dx dt \\
&\lesssim \frac{\lambda^{-1+}}{N_3^{0+}} \| (J_x I v_1) (J_x^{0+} I v_3) \|_{L_{t,x}^2} \| (J_x I v_2) (I v_4) \|_{L_{t,x}^2} \\
&\lesssim \frac{\lambda^{-1+}}{N_3^{0+}} \prod_{j=1}^4 \| I v_j \|_{X^{1, \frac{1}{2}-}}.
\end{aligned} \tag{5.24}$$

Hence, we have the estimate<sup>13</sup>

$$\mu[v] \left| \int_0^\delta \Lambda_4(K_4^1; v(t)) dt \right| \lesssim \lambda^{-2+} \| I v \|_{Z^1([0, \delta] \times \mathbb{T}_\lambda)}^6.$$

This decay rate is insufficient to reach the regularity index  $s = \frac{1}{2}$ . Since the bound of  $K_4^1$  is optimal and the available tools cannot yield a better estimate, we have to provide a second correction term that removes (at least) this case.

**5.3. The third generation modified energy.** We refine further the choice of modified energy for the  $I$ -method as a refinement of  $\mathcal{E}^2$  of the form

$$\mathcal{E}^3[v] := \mathcal{E}^2[v] + \Lambda_6(\sigma_6; v) + i\mu[v] \Lambda_4(\widetilde{\sigma}_4; v). \tag{5.25}$$

In the same manner as above, we are lead to define the “correction” term  $\sigma_6$  by imposing  $M_6 + \sigma_6 \alpha_6 = 0$ . In contrast to the situation of  $\alpha_4$  discussed above, the set on which  $\alpha_6$  vanishes is not small, in particular  $\alpha_6 = 0$  does not imply  $M_6 = 0$ . The idea around this is to define a region  $\Omega$  in the hyperplane  $\Gamma_6(\mathbb{T}_\lambda)$  referred to as *the non-resonant set of  $\sigma_6$*  where  $\alpha_6$  clearly does not vanish, but also with the property that on  $\Omega^c := \Gamma_6(\mathbb{T}_\lambda) \setminus \Omega$  we have satisfactory pointwise estimates on  $M_6^2$ . We can then take

$$\sigma_6 := -\frac{M_6^2}{\alpha_6} \cdot \mathbb{1}_\Omega, \tag{5.26}$$

where  $\mathbb{1}_\Omega$  denotes the characteristic function of the set  $\Omega$  which is defined in Subsection 5.4.

For the second correction term in (5.25), the situation is simpler (since  $\alpha_4 = 0$  implies  $K_4^1 = 0$ ) and we can define

$$\widetilde{\sigma}_4 := \frac{K_4^1}{\alpha_4} \tag{5.27}$$

when  $\alpha_4 \neq 0$ , and  $\widetilde{\sigma}_4 := 0$  when  $\alpha_4 = 0$ .

Using (2.55), we find that

$$\begin{aligned}
\frac{d}{dt} \mathcal{E}^3[v(t)] &= \Lambda_6(M_6^2 \cdot \mathbb{1}_{\Omega^c}) + \Lambda_8(M_8^2) + \Lambda_8(M_8^3) + \Lambda_{10}(M_{10}^3) \\
&\quad - i\mu[v] \left( \Lambda_4(K_4^1) + \Lambda_6(K_6^1) + \Lambda_6(K_6^2) + \Lambda_6(\widetilde{K}_6^3) + \Lambda_8(K_8^3) + \Lambda_8(\widetilde{K}_8^3) \right) \\
&\quad + \mu[v]^2 \Lambda_6(\widetilde{K}_6^4)
\end{aligned} \tag{5.28}$$

<sup>13</sup>. In the region where we have the refined estimate  $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2 N_1 N_3$ , one obtains the pre-factor  $N^{-1+} \lambda^{-2+}$  in (5.24).

where the additional terms (i.e. the ones corresponding to the two correction terms  $\sigma_6$  and  $\widetilde{\sigma}_4$ ) are given by

$$M_8^3 := -i \sum_{j=1}^6 \mathbb{X}_j^2(\sigma_6) k_{j+1}, \quad (5.29)$$

$$K_8^3 := \sum_{j=1}^6 \mathbb{X}_j^2(\sigma_6), \quad (5.30)$$

$$M_{10}^3 := \frac{i}{2} \sum_{j=1}^6 (-1)^{j+1} \mathbb{X}_j^4(\sigma_6), \quad (5.31)$$

$$\widetilde{K}_6^3 := i \sum_{j=1}^4 \mathbb{X}_j^2(\widetilde{\sigma}_4) k_{j+1}, \quad (5.32)$$

$$\widetilde{K}_6^4 := \sum_{j=1}^4 \mathbb{X}_j^2(\widetilde{\sigma}_4), \quad (5.33)$$

$$\widetilde{K}_8^3 := \frac{i}{2} \sum_{j=1}^4 (-1)^j \mathbb{X}_j^4(\widetilde{\sigma}_4). \quad (5.34)$$

**5.4. A non-resonant set for  $\alpha_6$ .** We now turn to describing the set  $\Omega$ , as it was introduced in [31]. With the simplifying assumptions of Remark 5.2 in place, let us analyze the expression

$$i\alpha_6 = k_1^2 - k_2^2 + k_3^2 - k_4^2 + k_5^2 - k_6^2.$$

If precisely two frequencies have sizes above the threshold  $N$ , we distinguish the following two cases.

**Case 1.** If the largest two frequencies have the same parity, then clearly  $|\alpha_6| \gtrsim N_1^2$ . The corresponding non-resonant region is defined to be

$$\Omega_1 := \{\mathbf{k} \in \Upsilon_6(\mathbb{T}_\lambda) : |k_1| \sim |k_3| \gtrsim N \gg N_3\}. \quad (5.35)$$

This definition is just slightly different from the analogous one in [31, Section 3] and it does not affect the estimates.

**Case 2.** If the largest two frequencies have opposite parity, say  $k_1$  and  $k_2$ , then on  $\Gamma_6(\mathbb{T}_\lambda)$  it must be that  $k_1 = -k_2 + O(N_3)$  and

$$i\alpha_6 = k_{12}(k_1 - k_2) + O(N_3^2).$$

While  $k_1 - k_2 = O(N_1)$ , it is possible to have  $k_{12} = 0$  and  $\alpha_6 = 0$ . Even if the latter does not happen, a too weak lower bound on  $\alpha_6$  renders an insufficiently good upper bound on  $M_8^3$  (one of the multipliers that involve  $\sigma_6 = -\frac{M_6^2}{\alpha_6}$ , see (5.29)). As in [31], we consider the following subregion

$$\Omega_2 := \left\{ \mathbf{k} \in \Upsilon_6(\mathbb{T}_\lambda) : |k_1| \sim |k_2| \gtrsim N \gg N_3 \text{ and } |k_{12}| \gtrsim \left( \frac{N_3}{N_1} \right)^{\frac{1}{2}} N_3 \right\}. \quad (5.36)$$

**Remark 5.13.** Notice that in the case  $|k_1| \sim |k_2| \gtrsim N \gg N_3$ , we have  $|k_{12}| = |k_{3456}| \lesssim N_3$ . On the other hand, by looking to ensure  $\alpha_6 \neq 0$ , the natural bound to impose is  $|k_{12}| \gtrsim \frac{N_3^2}{N_1}$ . However, while the latter gives a better bound on the remaining part  $M_6^2 \mathbf{1}_{\Omega^c}$  of  $\frac{d}{dt} \mathcal{E}^3$ , it

does not allow for a satisfactory bound on the correction multiplier  $\sigma_6$  (which appears, for example, in  $M_8^3$ ). At the other extreme, correcting only in the region  $|k_{12}| \sim N_3$  does not produce a small enough bound on  $M_6^2 \mathbf{1}_{\Omega^c}$ . We would like to point out that (here, as well as in the Euclidean setting [31]), the choice of  $\frac{1}{2}$  in the exponent is not essential, as any lower bound of the form

$$|k_{12}| \gtrsim \left( \frac{N_3}{N_1} \right)^\theta N_3$$

(for some  $0 < \theta < 1$ ) produces the extra  $N^{-\theta}$  decay factor needed to reach  $s = \frac{1}{2}$ .

**Case 3.** Finally, since the decay factors in the estimate of  $\Lambda_6(M_6^2)$ -term were also critical in the case  $N_3 \gtrsim N \gg N_4$  (see Case 2 in the proof of Proposition 5.9), we need to correct for it in this region as well. When three frequency sizes are much larger than the remaining frequency sizes,  $\alpha_6$  does not vanish as we have  $|\alpha_6| \gtrsim N_3^2$ . Therefore, we define

$$\Omega_3 := \{\mathbf{k} \in \Upsilon_6(\mathbb{T}_\lambda) : N_3 \gg N_4\} \quad (5.37)$$

We point out that the correction is deliberately intended for the larger region  $N_3 \gg N_4$  (i.e.  $\Omega_3$ ) rather than  $N_3 \gtrsim N \gg N_4$ , since on  $\Omega_3$  we have

$$|k_1^* + k_2^*| = |k_3^* + k_4^* + k_5^* + k_6^*| \sim N_3 \gtrsim \left( \frac{N_3}{N_1} \right)^{\frac{1}{2}} N_3. \quad (5.38)$$

Correcting for  $M_6^2$  in these three subregions of  $\Upsilon_6(\mathbb{T}_\lambda)$  is enough for our goal, hence we consider  $\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3$  to be the non-resonant set of  $\alpha_6$ , and in what follows we denote  $\Omega^c := \Upsilon_6(\mathbb{T}_\lambda) \setminus \Omega$ .

**5.5. Pointwise bounds on the multipliers (continued).** In this section we first recall the pointwise estimates obtained by Miao, Wu, and Xu [31], and then we establish the bounds needed to handle the second correction term in (5.25).

**Lemma 5.14.** [31, Corollary 4.1] *For  $M_6^2$  defined by (5.14) and  $\mathbf{k} \in \Gamma_6$ , we have:*

- (i) *if  $N_3 \ll N$ , then  $|M_6^2(\mathbf{k})| \lesssim N_1 |k_1^* + k_2^*| + N_3^2$ ;*
- (ii) *if  $N_3 \ll N$  and  $\mathbf{k} \in \Omega^c$ , then  $|M_6^2(\mathbf{k})| \lesssim N_1^{\frac{1}{2}} N_3^{\frac{3}{2}}$ .*

**Lemma 5.15.** [31, Lemma 4.9] *For  $\sigma_6$  defined by (5.26) and  $\mathbf{k} \in \Gamma_6(\mathbb{T}_\lambda)$ , we have:*

- (i)  $|\sigma_6(\mathbf{k})| \lesssim 1$ ;
- (ii) *if  $\mathbf{k} \in \Omega_1 \cap \{N_3 \ll N\}$ , then  $|\sigma_6(\mathbf{k})| \lesssim \frac{N_3}{N_1}$ .*

**Lemma 5.16.** [31, Proposition 4.3] *For  $M_8^3$  defined by (5.29) and  $\mathbf{k} \in \Gamma_8$ , we have:*

- (i)  $|M_8^3(\mathbf{k})| \lesssim N_1$ ;
- (ii) *if  $N_3 \ll N$ , then  $|M_8^3(\mathbf{k})| \lesssim N_1^{\frac{1}{2}} N_3^{\frac{1}{2}}$ .*

Also, as direct consequences of the above Lemma 5.15, we have the same bounds for  $K_8^3$  and  $M_{10}^3$  (see (5.30) and (5.31)) as for  $\sigma_6$ . Finally, we provide the pointwise estimates corresponding to the second correction term in (5.25).

**Lemma 5.17.** *For  $\widetilde{\sigma}_4$  defined by (5.27) and  $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$ , we have:*

- (i)  $|\widetilde{\sigma}_4(\mathbf{k})| \lesssim m(N_1)^2 N_1$ ;
- (ii) *if  $N_3 \ll N$ , then  $|\widetilde{\sigma}_4(\mathbf{k})| \lesssim m(N_1)^2$ .*

*Proof.* Let  $\beta_4$  denote the numerator in (5.27). We have the crude estimate  $|\beta_4| \lesssim m(N_1)^2 N_1^2$ , and note that either  $\alpha_4 = 0$  (in which case  $\widetilde{\sigma}_4 = 0$ ) or  $|\alpha_4| \geq 2N_1$ . Depending on the parity of the largest two frequencies, we distinguish two cases.

If  $k_1^* = k_1$  and  $k_2^* = k_3$ , then  $|\alpha_4| \sim |k_{12}||k_{14}| \sim N_1^2$  and  $|\beta_4| \sim m(N_1)^2 N_1^2$ .

If  $k_1^* = k_1$  and  $k_2^* = k_2$ , then  $|\alpha_4| \sim N_1|k_{34}|$ ,  $k_1$  and  $k_2$  have opposite signs and by the mean value theorem, we have

$$\begin{aligned} |\beta_4| &\leq |m(k_1)^2 k_1^2 - m(-k_2)^2 (-k_2)^2| + |k_{34}| \cdot |k_3 - k_4| \\ &\leq |k_{12}| \cdot |(m(\xi)^2 \xi^2)'| + |k_{34}| \cdot |k_3 - k_4|, \end{aligned}$$

where  $|\xi| \sim N_1$  and thus

$$\left| \frac{d}{d\xi} (m(\xi)^2 \xi^2) \right| \sim |m(\xi)^2 \xi| \sim m(N_1)^2 N_1.$$

Since

$$|k_3 - k_4| \lesssim N_3 \ll N \lesssim m(N_1)^2 N_1,$$

we get

$$|\beta_4| \lesssim m(N_1)^2 N_1 |k_{34}|$$

and the conclusion follows.  $\square$

Consequently, by simply referring to their definitions in (5.33) and (5.34), we also have the same bounds for  $\widetilde{K}_6^4$  and  $\widetilde{K}_8^3$ , respectively, as for  $\widetilde{\sigma}_4$ . In the same manner, we have the following lemma.

**Lemma 5.18.** *For  $\widetilde{K}_6^3$  defined by (5.32) and  $\mathbf{k} \in \Gamma_6(\mathbb{T}_\lambda)$ , we have:*

- (i)  $|\widetilde{K}_6^3(\mathbf{k})| \lesssim m(N_1)^2 N_1^2$ ;
- (ii) if  $N_3 \ll N$ , then  $|\widetilde{K}_6^3(\mathbf{k})| \lesssim m(N_1)^2 N_1$ .

## 6. ALMOST CONSERVATION ESTIMATES FOR THE THIRD GENERATION MODIFIED ENERGY

The scope of this section is to show that for a smooth solution  $v$  of (2.43), the possible increase of  $\mathcal{E}^3[v(\cdot)]$  can be made arbitrary small by appropriately choosing the parameters  $N$  and  $\lambda$ , i.e. that we have an estimate of the form

$$|\mathcal{E}^3[v(\delta)] - \mathcal{E}^3[v(0)]| \lesssim N^{-\gamma} \lambda^{-\kappa} \quad (6.1)$$

for some  $\gamma, \kappa > 0$ .<sup>14</sup> On the right hand side we use (powers of) the  $Z^1$ -norm of  $Iv$  who, we recall, lives on the scaled spatial domain  $\mathbb{T}_\lambda$  and whose energy on frequencies  $\gtrsim N$  is damped by the operator  $I$ .

We decompose the solution  $v$  using the Littlewood-Paley projectors in spatial frequencies:

$$v = \sum_{j=0}^{\infty} P_{2^j} v \quad , \quad \widehat{P_{2^j} v}(\tau, k) = \mathbf{1}_{I_j}(n) \widehat{v}(\tau, k),$$

---

14. The powers  $\gamma, \kappa$  are responsible for the level of regularity at which the global existence via the  $I$ -method is obtained. Subsequent iterations of the  $I$ -method aim at finding a functional that can provide good enough decay rates in order to reach  $s = \frac{1}{2}$ . Eventually, one approximates an  $H^s(\mathbb{T})$ -solution  $v$  of (2.43) by a sequence of smooth solutions  $(v_n)_n$  and then (6.1) is derived for  $v$ .

where  $I_0 := \{k \in \mathbb{Z}_\lambda : |k| < 1\}$  and  $I_j := \{k \in \mathbb{Z}_\lambda : 2^{j-1} \leq |k| < 2^j\}$  for  $j \geq 1$ . By the fundamental theorem of calculus, the proof of (6.1) reduces to estimating expressions of the form

$$\int_0^\delta \Lambda_n(M_n; v(t)) dt$$

corresponding to the multipliers  $M_n$  that appear in (5.28). It is enough<sup>15</sup> to obtain estimates for

$$\int_{\mathbb{R}} \mathbf{1}_{[0,\delta]}(t) \Lambda_n(M_n; v_1(t), \dots, v_n(t)) dt, \quad (6.2)$$

where each  $v_j$  has Fourier support in the band  $\{(\tau, k) : |k| \sim N_j\}$ , with  $N_j \sim |I_j|$ . If  $N_j \ll N$  for all  $j$ , the multiplier  $M_n$  vanishes, hence we assume  $N_1 \sim N_2 \gtrsim N$  (see Remark 5.1). Due to Remark 5.2, we can also assume that  $N_1 \geq N_2 \geq \dots \geq N_n$ .

Regarding the sharp time-cutoff, we note that in each case, we are able to place at least a few factors in the  $X^{1, \frac{1}{2}-}$ -norm (rather than in the  $Z^1$ -norm) and since we know that

$$\|\mathbf{1}_{[0,\delta]}(t)\|_{H_t^{\frac{1}{2}-}} \lesssim \delta^{0+},$$

by Lemma 2.4, we have

$$\|\mathbf{1}_{[0,\delta]} I v\|_{X^{1, \frac{1}{2}-}} \lesssim \delta^{0+} \|I v\|_{X^{1, \frac{1}{2}}}. \quad (6.3)$$

Therefore, in proving the results of this section, we are concerned with estimates of the form

$$\int_{\mathbb{R}} \Lambda_n(M_n; v_1(t), \dots, v_n(t)) dt \lesssim N^{-\gamma} \lambda^{-\kappa} \prod_{j=1}^n \|I v_j\|_{Z^1(\mathbb{R} \times \mathbb{T}_\lambda)}, \quad (6.4)$$

where  $v_j = P_{N_j} v_j$  for all  $j$ .

Before starting to prove estimates of the form (6.4) for each term that appears in (5.28), we make some further reductions common to all of them.

**Remark 6.1.** Since the norms on the right hand side of (6.4) depend on  $|\widehat{v_j}|$ , for the sake of simplified writing, we assume that all  $\widehat{v_j}$ 's are real valued and non-negative.

**Remark 6.2.** To ensure summability over all dyadics  $N_1 \geq N_2 \geq \dots \geq N_n$ , we can most of the times obtain a factor of  $1/N_1^{0+}$  on the right hand side above since then

$$\frac{1}{N_1^{0+}} \prod_{j=1}^n \|I P_{N_j} v_j\|_{Z^1(\mathbb{R} \times \mathbb{T}_\lambda)} \lesssim \left( \prod_{j=1}^n \frac{1}{N_j^{0+}} \right) \|I v_j\|_{Z^1(\mathbb{R} \times \mathbb{T}_\lambda)}^n,$$

and the summation (first over  $N_n$ , lastly over  $N_1$ ) is straightforward. However, having  $L_{\tau,k}^2$ -based norms on the largest two frequency factors  $I v_1$  and  $I v_2$  allows one to relax the summability factor to  $1/N_3^{0+}$  in the region  $N_1 \sim N_2$ . This essentially follows from an application of Cauchy-Schwarz inequality. Indeed, suppose that we have

$$|\mathcal{L}(P_{N_1} v_1, P_{N_2} v_2)| \leq A \|I P_{N_1} v_1\|_{X^{1, \frac{1}{2}}} \|I P_{N_2} v_2\|_{X^{1, \frac{1}{2}}}$$

for the bilinear functional  $\mathcal{L}$  defined by fixing  $v_3, \dots, v_n$  in the left hand side of (6.4). Let  $N_1 = 2^{j_1}$  and  $N_2 = 2^{j_2}$ . Summing over the pair of dyadic numbers  $(N_1, N_2)$  in the

<sup>15</sup> Indeed, one can take the functions  $v_j$  such that the time restrictions  $v_j|_{[0,\delta]} = P_j v$  and  $\|v_j\|_{Z^1(\mathbb{R} \times \mathbb{T}_\lambda)} \leq \|P_j v\|_{Z^1([0,\delta] \times \mathbb{T}_\lambda)} + \varepsilon$  for odd  $j$ 's, and similarly with  $P_j \bar{v}$  for even  $j$ 's. Eventually one takes  $\varepsilon \rightarrow 0$  to obtain the estimate.

region  $N_1 \sim N_2$  amounts to summing over the pair of integers  $(j_1, j_2)$  with  $|j_1 - j_2| \leq 4$ .<sup>16</sup> Therefore

$$\begin{aligned} \sum_{N_1 \sim N_2} |\mathcal{L}(P_{N_1} v_1, P_{N_2} v_2)| &\leq A \left( \sum_{j_1 \in \mathbb{Z}} \|P_{2^{j_1}} w_1\|_{L_{t,x}^2}^2 \right)^{\frac{1}{2}} \left( \sum_{j_2 \in \mathbb{Z}, |j_2 - j_1| \leq 4} \|P_{2^{j_2}} w_2\|_{L_{t,x}^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim A \|w_1\|_{L_{t,x}^2} \|w_2\|_{L_{t,x}^2}, \end{aligned}$$

where we have taken  $w_j$  to be defined by  $\widehat{w_j}(\tau, k) = m(k) \langle k \rangle \langle \tau + k^2 \rangle^{\frac{1}{2}} \widehat{v_j}(\tau, k)$ .

With these reduction remarks at hand, we can proceed to the proof of almost conservation estimates. We denote by  $J_x$  the Bessel potential operator in the space variable, i.e.  $\widehat{J_x f}(k) = \langle k \rangle \widehat{f}(k)$ . For simplicity,  $\int_*$  and  $\int_{**}$  stand for integration on  $\Gamma_6(\mathbb{R})$  and on  $\Gamma_6(\mathbb{T}_\lambda)$ , respectively (see Section 2.5).

**Lemma 6.3.** *Let  $s \geq \frac{1}{2}$  and  $\delta > 0$ . For  $M_6^2$  defined by (5.14), and  $\Omega^c$  as in Subsection 5.4, we have the estimate*

$$\left| \int_0^\delta \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v(t)) dt \right| \lesssim N^{-\frac{3}{2}+} \lambda^{-1+} \delta^{0+} \|Iv\|_{Z^1([0,\delta] \times \mathbb{T}_\lambda)}^6. \quad (6.5)$$

*Proof.* We distinguish several subregions of  $\Omega^c$ , but first note that for all  $\mathbf{k} \in \Upsilon_6(\mathbb{T}_\lambda) \setminus \Omega_3$  we have  $N_3 \sim N_4$ .

**Case 1:**  $N_1 \sim N_2 \gtrsim N \gg N_3$ . Note that  $m(N_j) = 1$  for  $j \geq 3$ , and  $m(N_1)^2 N_1 \gtrsim N$ . By Lemma 5.14, we have  $|M_6^2 \mathbf{1}_{\Omega^c}| \lesssim N_1^{\frac{1}{2}} N_3^{\frac{3}{2}}$  and by using (2.51), we get

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6) dt &\lesssim \int_* \int_{**} \frac{1}{m(N_1)^2 N_1^{\frac{3}{2}} N_3^{\frac{1}{2}} \prod_{j=5,6} \langle k_j \rangle} \prod_{j=1}^6 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-\frac{3}{2}+}}{N_1^{0+}} \int_* \int_{**} \frac{1}{\langle k_5 \rangle \langle k_6 \rangle} \prod_{j=1}^6 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-\frac{3}{2}+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x I v_3)(J_x I v_2)(J_x I v_4)(I v_5)(I v_6) dx dt \\ &\lesssim \frac{N^{-\frac{3}{2}+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L_{t,x}^2} \|(J_x I v_2)(J_x I v_4)\|_{L_{t,x}^2} \prod_{j=5,6} \|I v_j\|_{L_{t,x}^\infty}. \end{aligned}$$

By (2.25) and (2.10), we thus get

$$\int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6) dt \lesssim \frac{N^{-\frac{3}{2}+} \lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^4 \|I v_j\|_{X^{1,\frac{1}{2}}} \prod_{j=5,6} \|I v_j\|_{Y^{\frac{1}{2}+},0}.$$

The case  $N_3 \gtrsim N \gg N_4$  is vacuous on  $\Omega^c$  and thus the next case we have to consider is the one in which precisely four of the frequencies have sizes larger than the threshold  $N$ .

**Case 2:**  $N_4 \gtrsim N \gg N_5$ . We also have  $N_1 \sim N_2$ ,  $N_3 \sim N_4$  and for  $j = 3, 4$ ,

$$m(N_j) N_j = N^{1-} \left( \frac{N_j}{N} \right)^{s-} N_j^{0+} \gtrsim N^{1-} N_j^{0+}. \quad (6.6)$$

16. For  $n \leq 10$ , on  $\Gamma_n(\mathbb{T}_\lambda)$  we have  $\frac{1}{9} N_2 \leq N_1 \leq 9 N_2$ , so there is a universal upper bound on  $|j_1 - j_2|$ .



By using the crude estimate  $|M_6^2| \lesssim m(N_1)^2 N_1^2$  of Lemma 5.4, we estimate

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6) dt &\lesssim \int_* \int_* \frac{1}{m(N_3)^2 N_3^2 \langle k_5 \rangle \langle k_6 \rangle} \prod_{j=1}^6 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-2+}}{N_3^{0+}} \int_* \frac{1}{\langle k_5 \rangle \langle k_6 \rangle} \prod_{j=1}^6 \widehat{J_x I v_j}. \end{aligned} \quad (6.7)$$

We now discuss two subcases.

**Subcase 2.1.** If  $N_3 \sim N_1$ , since  $N_5 \ll N_4$ , two out of the four frequencies  $k_1, k_2, k_3, k_4$  must have opposite signs, say  $k_1$  and  $k_2$ . Therefore  $v_1$  and  $v_2$  are separated in frequency and we use the bilinear estimate (2.25), and together with the  $L^4$ -Strichartz estimate (2.18), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6) dt &\lesssim \frac{N^{-2+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_2)\|_{L_{t,x}^2} \prod_{j=3,4} \|J_x I v_j\|_{L_{t,x}^4} \prod_{j=5,6} \|I v_j\|_{L_{t,x}^\infty} \\ &\lesssim \frac{N^{-2+} \lambda^{-\frac{1}{2}+}}{N_1^{0+}} \prod_{j=1}^4 \|J_x I v_j\|_{X^{0, \frac{1}{2}}} \prod_{j=5,6} \|I v_j\|_{Y^{\frac{1}{2}+, 0}}. \end{aligned}$$

**Subcase 2.2.** If  $N_3 \ll N_1$ , then we apply the bilinear estimate (2.25) twice and get

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, v_2, \dots, v_6) dt &\lesssim \frac{N^{-2+}}{N_3^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L_{t,x}^2} \|(J_x I v_2)(J_x I v_4)\|_{L_{t,x}^2} \prod_{j=5,6} \|I v_j\|_{L_{t,x}^\infty} \\ &\lesssim \frac{N^{-2+} \lambda^{-1+}}{N_3^{0+}} \prod_{j=1}^4 \|J_x I v_j\|_{X^{0, \frac{1}{2}}} \prod_{j=5,6} \|I v_j\|_{Y^{\frac{1}{2}+, 0}}. \end{aligned}$$

**Case 3:**  $N_5 \gtrsim N$ . We use (6.6) for  $j = 3, 4, 5$ ,  $m(k_6) \langle k_6 \rangle^{\frac{1}{2}} \gtrsim 1$ , and  $N_5 \geq N_6$  to deduce

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6) dt &\lesssim \int_* \int_{**} \frac{1}{\prod_{j=3}^6 m(k_j) \langle k_j \rangle} \prod_{j=1}^6 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-3+}}{N_3^{0+}} \int_* \int_{**} \widehat{J_x I v_1} \widehat{J_x I v_2} \prod_{j=3}^5 \left( \frac{\lambda^{0+}}{\langle k_j \rangle^{0+}} \widehat{J_x I v_j} \right) \left( \frac{1}{\langle k_6 \rangle^{\frac{1}{2}+}} \widehat{J_x I v_6} \right) \\ &\lesssim \frac{N^{-3+} \lambda^{0+}}{N_3^{0+}} \prod_{j=1,2} \|J_x I v_j\|_{L_{t,x}^4} \prod_{j=3}^5 \|J_x^{1-} I v_j\|_{L_{t,x}^6} \|J_x^{\frac{1}{2}-} I v_6\|_{L_{t,x}^\infty}. \end{aligned}$$

The factors  $\lambda^{0+}$  above appear due to the application of (2.3). By using the Strichartz estimates (2.18) and (2.21), as well as the embedding (2.10), we have

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_6(M_6^2 \mathbf{1}_{\Omega^c}; v_1, \dots, v_6) dt &\lesssim \frac{N^{-3+} \lambda^{0+}}{N_3^{0+}} \prod_{j=1,2} \|J_x I v_j\|_{X^{0, \frac{3}{8}}} \prod_{j=3}^5 \|J_x^{1-} I v_j\|_{X^{0+, \frac{1}{2}}} \|J_x^{\frac{1}{2}-} I v_6\|_{Y^{\frac{1}{2}+, 0}} \\ &\lesssim \frac{N^{-3+} \lambda^{0+}}{N_3^{0+}} \prod_{j=1}^5 \|I v_j\|_{X^{1, \frac{1}{2}}} \|I v_6\|_{Y^{1, 0}}. \end{aligned}$$

Since in Section 8 we choose  $\lambda, N$  such that  $1 \leq \lambda \leq N$  (for  $s \geq \frac{1}{2}$ ), in the second and third cases we have faster decaying factors than in Case 1.  $\square$

**Lemma 6.4.** *Let  $s \geq \frac{1}{2}$  and  $\delta > 0$ . For  $M_8^3$  defined by (5.29), we have the estimate*

$$\left| \int_0^\delta \Lambda_8(M_8^3; v(t)) dt \right| \lesssim N^{-\frac{3}{2}+} \lambda^{-1+} \delta^{0+} \|Iv\|_{Z^1([0,\delta] \times \mathbb{T}_\lambda)}^8. \quad (6.8)$$

The same estimate holds if  $M_8^3$  is replaced by  $M_8^2$ .

*Proof.* By Lemma 5.16, we have  $|M_8^3(\mathbf{k})| \lesssim N_1$  for all  $\mathbf{k} \in \Gamma_8(\mathbb{T}_\lambda)$ , and if  $N_3 \ll N$ , then  $|M_8^3(\mathbf{k})| \lesssim N_1^{\frac{1}{2}} N_3^{\frac{1}{2}}$ .

We distinguish three cases and in all of them we use that  $m(N_1)^2 N_1 \gtrsim N$ , and when  $N_3 \gtrsim N$ ,  $m(N_3) N_3 \gtrsim N^{1-N_3^{0+}}$  as in (6.6).

**Case 1:**  $N_1 \sim N_2 \gtrsim N \gg N_3$ . We have

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_8(M_8^3; v_1, \dots, v_8) dt &\lesssim \int_* \int_{**} \frac{1}{m(N_1)^2 N_1^{\frac{3}{2}} N_3^{\frac{1}{2}} \prod_{j=4}^8 \langle k_j \rangle} \prod_{j=1}^8 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-\frac{3}{2}+}}{N_1^{0+}} \int_* \int_{**} (\widehat{J_x I v_1} \widehat{J_x I v_3}) (\widehat{J_x I v_2} \widehat{J_x I v_4}) \prod_{j=5}^8 \widehat{I v_j} \\ &\lesssim \frac{N^{-\frac{3}{2}+}}{N_1^{0+}} \| (J_x I v_1) (J_x I v_3) \|_{L_{t,x}^2} \| (J_x I v_2) (J_x I v_4) \|_{L_{t,x}^2} \prod_{j=5}^8 \| I v_j \|_{L_{t,x}^\infty} \\ &\lesssim \frac{N^{-\frac{3}{2}+} \lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^4 \| I v_j \|_{X^{1,\frac{1}{2}}} \prod_{j=5}^8 \| I v_j \|_{Y^{\frac{1}{2}+,0}}. \end{aligned}$$

**Case 2:**  $N_3 \gtrsim N \gg N_4$ . Here, we get

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_8(M_8^3; v_1, \dots, v_8) dt &\lesssim \int_* \int_{**} \frac{1}{m(N_1)^2 N_1 m(N_3) N_3 \prod_{j=4}^8 \langle k_j \rangle} \prod_{j=1}^8 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-2+}}{N_3^{0+}} \int_* \int_{**} (\widehat{J_x I v_1} \widehat{J_x I v_4}) (\widehat{J_x I v_2}) (\widehat{J_x I v_3}) \prod_{j=5}^8 \widehat{I v_j} \\ &\lesssim \frac{N^{-2+}}{N_3^{0+}} \| (J_x I v_1) (J_x I v_4) \|_{L_{t,x}^2} \| J_x I v_2 \|_{L_{t,x}^4} \| J_x I v_3 \|_{L_{t,x}^4} \prod_{j=5}^8 \| I v_j \|_{L_{t,x}^\infty} \\ &\lesssim \frac{N^{-2+} \lambda^{-\frac{1}{2}+}}{N_3^{0+}} \prod_{j=1}^4 \| I v_j \|_{X^{1,\frac{1}{2}}} \prod_{j=5}^8 \| I v_j \|_{Y^{\frac{1}{2}+,0}}. \end{aligned}$$

**Case 3:**  $N_4 \gtrsim N$ . In this case, we additionally have that  $m(N_4) N_4 \gtrsim N$ . For  $5 \leq j \leq 8$ , since  $m(k_j) \langle k_j \rangle^{\frac{1}{2}} \gtrsim 1$ , by taking into account (2.3), we have

$$N_3^{0+} m(k_j) \langle k_j \rangle \gtrsim \lambda^{0-} \langle k_j \rangle^{\frac{1}{2}+}. \quad (6.9)$$

Thus, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_8(M_8^3; v_1, \dots, v_8) dt &\lesssim \frac{N^{-3+}}{N_3^{0+}} \int_* \int_{**} \prod_{j=1}^4 \widehat{J_x I v_j} \prod_{j=5}^8 \frac{\lambda^{0+}}{\langle k_j \rangle^{\frac{1}{2}+}} \widehat{J_x I v_j} \\
&\lesssim \frac{N^{-3+} \lambda^{0+}}{N_3^{0+}} \prod_{j=1}^4 \|J_x I v_j\|_{L_{t,x}^4} \prod_{j=5}^8 \|J_x^{\frac{1}{2}-} I v_j\|_{L_{t,x}^\infty} \\
&\lesssim \frac{N^{-3+} \lambda^{0+}}{N_3^{0+}} \prod_{j=1}^4 \|J_x I v_j\|_{X^{0, \frac{3}{8}}} \prod_{j=5}^8 \|J_x^{\frac{1}{2}-} I v_j\|_{Y^{\frac{1}{2}+, 0}} \\
&\lesssim \frac{N^{-3+} \lambda^{0+}}{N_3^{0+}} \prod_{j=1}^4 \|I v_j\|_{X^{1, \frac{1}{2}}} \prod_{j=5}^8 \|I v_j\|_{Y^{1, 0}}.
\end{aligned}$$

We recall that for the multiplier  $M_8^2$  we have better bounds than for  $M_8^3$  (see Lemma 5.4 and Lemma 5.14), hence it is enough to consider only the latter.  $\square$

**Lemma 6.5.** *Let  $s \geq \frac{1}{2}$  and  $\delta > 0$ . For  $M_{10}^3$  defined by (5.31), we have the estimate*

$$\left| \int_0^\delta \Lambda_{10}(M_{10}^3; v(t)) dt \right| \lesssim N^{-2+} \lambda^{-1+} \delta^{0+} \|I v\|_{Z^1([0, \delta] \times \mathbb{T}_\lambda)}^{10}. \quad (6.10)$$

*Proof.* By (5.31) and Lemma 5.15, we have  $|M_{10}^3(\mathbf{k})| \lesssim 1$  and thus we gain the factor  $N^{-2+}$  from  $m(N_j)N_j \gtrsim N^{1-}N_j^{0+}$ ,  $j = 1, 2$ . For additional decaying factors, it is enough to discuss two cases.

**Case 1:**  $N_2 \gtrsim N \gg N_3$ . We have

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_{10}(M_{10}^3; v_1, \dots, v_{10}) dt &\lesssim \frac{N^{-2+}}{N_1^{0+}} \int_* \int_{**} (\widehat{J_x I v_1} \widehat{J_x I v_3}) (\widehat{J_x I v_2} \widehat{J_x I v_4}) \prod_{j=5}^{10} \widehat{I v_j} \\
&\lesssim \frac{N^{-2+}}{N_1^{0+}} \|(\widehat{J_x I v_1}) (\widehat{J_x I v_3})\|_{L_{t,x}^2} \|(\widehat{J_x I v_2}) (\widehat{J_x I v_4})\|_{L_{t,x}^2} \prod_{j=5}^{10} \|I v_j\|_{L_{t,x}^\infty} \\
&\lesssim \frac{N^{-2+} \lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^4 \|I v_j\|_{X^{1, \frac{1}{2}}} \prod_{j=5}^{10} \|I v_j\|_{Y^{\frac{1}{2}+, 0}}
\end{aligned}$$

**Case 2:**  $N_3 \gtrsim N$ . In this case, we additionally have  $m(N_3)N_3 \gtrsim N$ . Also, we use  $m(k_4)\langle k_4 \rangle \gtrsim 1$ , and  $m(k_j)\langle k_j \rangle^{\frac{1}{2}} \gtrsim 1$  for  $5 \leq j \leq 10$ . By using  $1/N_1^\varepsilon \leq \prod_{j=5}^{10} 1/N_j^{\varepsilon/6}$ , we get

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_{10}(M_{10}^3; v_1, \dots, v_{10}) dt &\lesssim \frac{N^{-3+}}{N_1^{0+}} \prod_{j=1}^4 \|J_x I v_j\|_{L_{t,x}^4} \prod_{j=5}^{10} \|J_x^{\frac{1}{2}-} I v_j\|_{L_{t,x}^\infty} \\
&\lesssim \frac{N^{-3+}}{N_1^{0+}} \prod_{j=1}^4 \|I v_j\|_{X^{1, \frac{3}{8}}} \prod_{j=5}^{10} \|I v_j\|_{Y^{1, 0}}.
\end{aligned}$$

Note that in Case 2 (by discussing various subregions), we could provide at least an additional  $\lambda^{-\frac{1}{2}+}$  factor, but since  $N^{-1+} \lesssim \lambda^{-1+}$  and the decaying factor in Case 1 is optimal, we limit ourselves to the above estimate.  $\square$

For the remaining terms that appear in (5.28) (i.e. the ones due to the gauge transformation in the periodic setting), we have a decaying factor  $\lambda^{-1}$  thanks to the coupling coefficient  $\mu[v]$ . Indeed, by (2.11) and by using  $1 \leq m(k)\langle k \rangle$ , we have

$$\mu[v] = \frac{1}{2\pi\lambda} \|v\|_{L_t^\infty L_x^2}^2 \lesssim \lambda^{-1} \|J_x I v\|_{Y^{0,0}}^2 \leq \lambda^{-1} \|I v\|_{Z^1}^2.$$

**Lemma 6.6.** *Let  $s > 0$  and  $\delta > 0$ . For  $K_4^1$  as defined by (5.6), we have the estimate*

$$\left| \int_0^\delta \Lambda_4(K_4^1; v(t)) dt \right| \lesssim N^{-1+} \lambda^{-1+} \delta^{0+} \|I v\|_{X^{1, \frac{1}{2}}([0, \delta] \times \mathbb{T}_\lambda)}^4. \quad (6.11)$$

*Proof.* By Lemma 5.7 we have  $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2 N_1^2$  for all  $\mathbf{k} \in \Gamma_4(\mathbb{T}_\lambda)$ , and if  $N_3 \ll N$  then  $|K_4^1(\mathbf{k})| \lesssim m(N_1)^2 N_1 N_3$ . We need to discuss three cases.

**Case 1:**  $N_1 \sim N_2 \gtrsim N \gg N_3$ . Due to the refined estimate, we have

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_4(K_4^1; v_1, \dots, v_4) dt &\lesssim \int_* \int_{**} \frac{1}{N_1 \langle k_4 \rangle} \prod_{j=1}^4 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-1+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x I v_3)(J_x I v_2)(J_x I v_4) dx dt \\ &\lesssim \frac{N^{-1+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L_{t,x}^2} \|(J_x I v_2)(J_x I v_4)\|_{L_{t,x}^2} \\ &\lesssim \frac{N^{-1+} \lambda^{-1+}}{N_1^{0+}} \prod_{j=1}^4 \|I v_j\|_{X^{1, \frac{1}{2}}}. \end{aligned}$$

**Case 2:**  $N_3 \gtrsim N \gg N_4$ . By using (6.6), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_4(K_4^1; v_1, \dots, v_4) dt &\lesssim \int_* \int_{**} \frac{1}{m(N_3) N_3 m(k_4) \langle k_4 \rangle} \prod_{j=1}^4 \widehat{J_x I v_j} \\ &\lesssim \frac{N^{-1+}}{N_3^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x I v_2)(J_x I v_3)(v_4) dx dt. \end{aligned}$$

**Subcase 2.1.** If  $N_3 \sim N_1$ , then since  $N_3 \gg N_4$ , two out of the three frequencies  $k_1, k_2, k_3$  must have opposite signs, say  $k_1$  and  $k_2$ . Thus  $J_x I v_1$  and  $J_x I v_2$  are separated in frequency, and so are  $J_x I v_3$  and  $J_x I v_4$ . By also using  $m(k_4) \langle k_4 \rangle \gtrsim 1$ , we have

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_4(K_4^1; v_1, \dots, v_4) dt &\lesssim \frac{N^{-1+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_2)\|_{L_{t,x}^2} \|(J_x I v_3)(J_x I v_4)\|_{L_{t,x}^2} \\ &\lesssim \frac{N^{-1+} \lambda^{-1+}}{N_3^{0+}} \prod_{j=1}^4 \|I v_j\|_{X^{1, \frac{1}{2}}}. \end{aligned} \quad (6.12)$$

**Subcase 2.2.** If  $N_3 \ll N_1$ , then as in Case 1, we can clearly apply the bilinear estimate (2.25) to the  $L_{t,x}^2$ -norms of both  $(J_x I v_1)(J_x I v_3)$  and  $(J_x I v_2)(J_x I v_4)$  and obtain the same bound as in (6.12).

**Case 3:**  $N_4 \gtrsim N$ . We have  $m(N_j)N_j \gtrsim N^{-1+}N_j^{0+}$  for  $j = 3, 4$  and thus

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_4(K_4^1; v_1, \dots, v_4) dt &\lesssim \int_* \int_{**} \frac{1}{m(N_3)N_3 m(N_4)N_4} \prod_{j=1}^4 \widehat{J_x I v_j} \\
&\lesssim \frac{N^{-2+}}{N_3^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} (J_x I v_1)(J_x I v_2)(J_x I v_3)(J_x I v_4) dx dt \\
&\lesssim \frac{N^{-2+}}{N_3^{0+}} \prod_{j=1}^4 \|J_x I v_j\|_{L_{t,x}^4} \\
&\lesssim \frac{N^{-2+}}{N_3^{0+}} \prod_{j=1}^4 \|I v_j\|_{X^{1, \frac{3}{8}}}.
\end{aligned}$$

□

**Lemma 6.7.** *Let  $s \geq \frac{3}{8}$  and  $\delta > 0$ . For  $K_6^1$  defined by (5.7), we have the estimate*

$$\left| \int_0^\delta \Lambda_6(K_6^1; v(t)) dt \right| \lesssim N^{-2+} \delta^{0+} \|Iv\|_{X^{1, \frac{1}{2}}([0, \delta] \times \mathbb{T}_\lambda)}^6. \quad (6.13)$$

*Proof.* By Lemma 5.8, we have  $|K_6^1(\mathbf{k})| \lesssim 1$  for all  $\mathbf{k} \in \Gamma_6(\mathbb{T}_\lambda)$ . By using (2.52), (2.16) and (2.18), we estimate

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_6(K_6^1; v_1, \dots, v_6) dt &\lesssim \int_* \int_{**} \frac{1}{(m(N_1)N_1)^2} \prod_{j=1,2} \widehat{J_x I v_j} \prod_{j=3}^6 \widehat{v_j} \\
&\lesssim \frac{N^{-2+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} \prod_{j=1,2} J_x I v_j \prod_{j=3}^6 J_x^{\frac{5}{8}} I v_j dx dt \\
&\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1,2} \|J_x I v_j\|_{L_{t,x}^4} \prod_{j=3}^6 \|J_x^{\frac{5}{8}} I v_j\|_{L_{t,x}^8} \\
&\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1}^6 \|I v_j\|_{X^{1, \frac{3}{8}}}.
\end{aligned}$$

□

**Lemma 6.8.** *Let  $s \geq \frac{1}{2}$  and  $\delta > 0$ . For  $K_6^2$  defined by (5.16), we have the estimate*

$$\left| \int_0^\delta \Lambda_6(K_6^2; v(t)) dt \right| \lesssim N^{-1+} \lambda^{-1+} \delta^{0+} \|Iv\|_{Z^1([0, \delta] \times \mathbb{T}_\lambda)}^6. \quad (6.14)$$

*Proof.* By Lemma 5.8, we have  $|K_6^2| \lesssim m(N_1)^2 N_1$ .

**Case 1:**  $N_2 \gtrsim N \gg N_3$ . By using  $1 \lesssim m(k_j)\langle k_j \rangle$  for  $j = 3, 4$ , and  $\frac{1}{N_1^{0+}} \lesssim m(k_j)\langle k_j \rangle^{\frac{1}{2}-}$  for  $j = 5, 6$ , we estimate

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_6(K_6^2; v_1, \dots, v_6) dt &\lesssim \int_{*} \int_{**} \frac{1}{N_1} \prod_{j=1,2} \widehat{J_x I v_j} \prod_{j=3}^6 \widehat{v_j} \\
&\lesssim \frac{N^{-1+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} \prod_{j=1}^4 J_x I v_j \prod_{j=5,6} J_x^{\frac{1}{2}-} I v_j dx dt \\
&\lesssim \frac{N^{-1+}}{N_1^{0+}} \|(J_x I v_1)(J_x I v_3)\|_{L_{t,x}^2} \|(J_x I v_2)(J_x I v_4)\|_{L_{t,x}^2} \prod_{j=5,6} \|J_x^{\frac{1}{2}-} I v_j\|_{L_{t,x}^\infty} \\
&\lesssim \frac{N^{-1+\lambda^{-1+}}}{N_1^{0+}} \prod_{j=1}^4 \|I v_j\|_{X^{1,\frac{3}{8}}} \prod_{j=5,6} \|I v_j\|_{Y^{1,0}}.
\end{aligned}$$

**Case 2:**  $N_3 \gtrsim N$ . We make use of  $m(N_3)N_3 \gtrsim N$  and thus we get

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_6(K_6^2; v_1, \dots, v_6) dt &\lesssim \frac{N^{-2+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} \prod_{j=1}^3 J_x I v_j \prod_{j=4}^6 v_j dx dt \\
&\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1}^3 \|J_x I v_j\|_{L_{t,x}^4} \prod_{j=4}^6 \|v_j\|_{L_{t,x}^{12}} \\
&\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1}^3 \|I v_j\|_{X^{1,\frac{3}{8}}} \prod_{j=4}^6 \|v_j\|_{X^{\frac{5}{12}, \frac{5}{12}}}.
\end{aligned}$$

□

For the next lemma, we make the following remark. The proof follows identically in Case 1, but we only have  $|\widetilde{K}_6^3| \lesssim m(N_1)^2 N_1^2$  in Case 2. By splitting the discussion into the subcases  $N_3 \sim N_1$  and  $N_3 \ll N_1$  as in Case 2 of the proof of Lemma 6.3, we can provide at least an additional  $\lambda^{-\frac{1}{2}+}$  factor. Hence, we have:

**Lemma 6.9.** *Let  $s \geq \frac{1}{2}$  and  $\delta > 0$ . For  $\widetilde{K}_6^3$  defined by (5.16), we have the estimate*

$$\left| \int_0^\delta \Lambda_6(\widetilde{K}_6^3; v(t)) dt \right| \lesssim N^{-1+\lambda^{-\frac{1}{2}+}} \delta^{0+} \|I v\|_{Z^1([0,\delta] \times \mathbb{T}_\lambda)}^6. \quad (6.15)$$

The estimates for  $\int_0^\delta \Lambda_6(\widetilde{K}_6^4) dt$  and  $\int_0^\delta \Lambda_8(\widetilde{K}_8^3) dt$  follow identically to that of Lemma 6.8 above, since we have the same upper bound (see Lemma 5.17 and the subsequent comment).

**Lemma 6.10.** *Let  $s \geq \frac{5}{12}$  and  $\delta > 0$ . For  $K_8^3$  defined by (5.30), we have the estimate*

$$\left| \int_0^\delta \Lambda_8(K_8^3; v(t)) dt \right| \lesssim N^{-2+\delta^{0+}} \|I v\|_{X^{1,\frac{1}{2}}([0,\delta] \times \mathbb{T}_\lambda)}^8. \quad (6.16)$$

*Proof.* By Lemma 5.15, we have  $|K_8^3(\mathbf{k})| \lesssim 1$  for all  $\mathbf{k} \in \Gamma_8(\mathbb{T}_\lambda)$ . Hence, similarly to the proof of Lemma 6.7, we get

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_8(K_8^3; v_1, \dots, v_8) dt &\lesssim \frac{N^{-2+}}{N_1^{0+}} \int_{\mathbb{R}} \int_{\mathbb{T}_\lambda} \prod_{j=1,2} J_x I v_j \prod_{j=3}^8 J_x^{\frac{7}{12}} I v_j dx dt \\ &\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1,2} \|J_x I v_j\|_{L_{t,x}^4} \prod_{j=3}^8 \|J_x^{\frac{7}{12}} I v_j\|_{L_{t,x}^{12}} \\ &\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1}^8 \|I v_j\|_{X^{1, \frac{1}{2}}}. \end{aligned}$$

□

We put all the results of this section together in the following:

**Proposition 6.11.** *Let  $s \geq \frac{1}{2}$  and  $\delta > 0$ . Suppose  $v$  is a solution to (2.43) on  $[0, \delta]$ . For  $\mathcal{E}^3$  defined by (5.25), we have*

$$|\mathcal{E}^3[v(\delta)] - \mathcal{E}^3[v(0)]| \leq N^{-\frac{3}{2}+} \lambda^{-1+} \delta^{0+} P(\|Iv\|_{Z^1([0, \delta] \times \mathbb{T}_\lambda)}), \quad (6.17)$$

for some polynomial  $P$  with non-negative coefficients.

## 7. CONTROL OF THE ALMOST CONSERVED ENERGY AND OF THE ALMOST CONSERVED MOMENTUM

In this section we show that  $\mathcal{E}[Iv(t)]$  stays close to  $\mathcal{E}^3[v(t)]$  (which is very slowly varying in time) and that  $\mathcal{P}[Iv(t)]$  stays close to  $\mathcal{P}[v(t)] = \mathcal{P}[v_0]$ , at any time  $t$ . For the sake of efficiency, we adopt in the proofs below the reduction remarks from the previous section.

**Lemma 7.1.** *For  $\sigma_4$  defined by (5.10), we have*

$$|\Lambda_4(\sigma_4; f)| \lesssim N^{-1+} \|If\|_{H^1(\mathbb{T}_\lambda)}^4. \quad (7.1)$$

*Proof.* By Lemma 5.5, we have  $|\sigma_4(\mathbf{k})| \lesssim m(N_1)^2 N_1$  for all  $k \in \Gamma_4(\mathbb{T}_\lambda)$ . Then, by Hölder and Sobolev inequalities, and using  $\frac{1}{N_1^{0+}} \lesssim m(k_j) \langle k_j \rangle^{\frac{1}{2}-}$  for  $j = 3, 4$ , we have

$$\begin{aligned} \Lambda_4(\sigma_4; f_1, \dots, f_4) &\lesssim \int_{**} \frac{m(N_1)^2 N_1}{\prod_{j=1}^4 m(k_j) \langle k_j \rangle} \prod_{j=1}^4 \widehat{J_x I f_j} \\ &\lesssim \frac{1}{N_1} \int_{\mathbb{T}_\lambda} (J_x I f_1)(J_x I f_2)(J_x^{\frac{1}{2}-} I f_3)(J_x^{\frac{1}{2}-} I f_4) dx \\ &\lesssim \frac{N^{-1+}}{N_1^{0+}} \|J_x I f_1\|_{L_x^2} \|J_x I f_2\|_{L_x^2} \|J_x^{\frac{1}{2}-} I f_3\|_{L_x^\infty} \|J_x^{\frac{1}{2}-} I f_4\|_{L_x^\infty} \\ &\lesssim \frac{N^{-1+}}{N_1^{0+}} \prod_{j=1}^4 \|I f_j\|_{H_x^1}. \end{aligned}$$

□

The estimate for  $\Lambda_4(\widetilde{\sigma}_4; f)$  follows similarly since, by Lemma 5.17 (i), we have the same pointwise bound, that is  $|\widetilde{\sigma}_4(\mathbf{k})| \lesssim m(N_1)^2 N_1$ .

**Lemma 7.2.** *For  $\sigma_6$  defined by (5.26), we have*

$$|\Lambda_6(\sigma_6; f)| \lesssim N^{-2+} \|If\|_{H^1(\mathbb{T}_\lambda)}^6. \quad (7.2)$$

*Proof.* By Lemma 5.15, we have  $|\sigma_6(\mathbf{k})| \lesssim 1$  for all  $k \in \Gamma_6(\mathbb{T}_\lambda)$ . Similarly to the proof of Lemma 7.1, we have

$$\begin{aligned} \Lambda_6(\sigma_6; f_1, \dots, f_4) &\lesssim \frac{N^{-2+}}{N_1^{0+}} \int_{\mathbb{T}_\lambda} (J_x If_1)(J_x If_2) \prod_{j=3}^6 (J_x^{\frac{1}{2}-} If_j) dx \\ &\lesssim \frac{N^{-2+}}{N_1^{0+}} \prod_{j=1}^6 \|If_j\|_{H_x^1}. \end{aligned}$$

□

Hence, we proved that all the correction terms are small, and thus we obtain:

**Proposition 7.3.** *For  $\mathcal{E}$  and  $\mathcal{E}^3$  defined by (5.1) and (5.25), we have*

$$|\mathcal{E}[If] - \mathcal{E}^3[f]| \lesssim N^{-1+} \left( \|If\|_{H_x^1(\mathbb{T}_\lambda)}^4 + \|If\|_{H_x^1(\mathbb{T}_\lambda)}^6 \right), \quad (7.3)$$

for all  $f \in H_x^s(\mathbb{T}_\lambda)$ .

Next, we turn to the analysis of  $\mathcal{P}[I(\cdot)]$  for which, as in [13], we prove:

**Proposition 7.4.** *Let  $s \geq \frac{1}{2}$ . For  $\mathcal{P}$  defined by (3.7), we have*

$$|\mathcal{P}[If] - \mathcal{P}[f]| \lesssim N^{-1} \left( \|If\|_{H^1(\mathbb{T}_\lambda)}^2 + \|If\|_{H^1(\mathbb{T}_\lambda)}^4 \right), \quad (7.4)$$

for all  $f \in H_x^s(\mathbb{T}_\lambda)$ .

*Proof.* We have

$$|\mathcal{P}[If] - \mathcal{P}[f]| \leq \left| \operatorname{Im} \int_{\mathbb{T}_\lambda} (If \partial_x (\overline{If}) - f \partial_x \overline{f}) dx \right| + \frac{1}{2} \left| \int_{\mathbb{T}_\lambda} (|If|^4 - |f|^4) dx \right| \quad (7.5)$$

and we can estimate the two terms separately.

First, using integration by parts, we write

$$\begin{aligned} \operatorname{Im} \int_{\mathbb{T}_\lambda} (If \partial_x (\overline{If}) - f \partial_x \overline{f}) dx &= \operatorname{Im} \int_{\mathbb{T}_\lambda} If \partial_x (\overline{If} - \overline{f}) dx + \operatorname{Im} \int_{\mathbb{T}_\lambda} \partial_x \overline{f} (If - f) dx \\ &= \operatorname{Im} \int_{\mathbb{T}_\lambda} If \partial_x (\overline{If} - \overline{f}) dx + \operatorname{Im} \int_{\mathbb{T}_\lambda} f \partial_x (\overline{If} - \overline{f}) dx \\ &= \operatorname{Im} \int_{\mathbb{T}_\lambda} (If + f) \partial_x (\overline{If} - \overline{f}) dx. \end{aligned}$$

Notice that  $I - \operatorname{Id} = P_{\text{hi}}(I - \operatorname{Id})$ , where  $\operatorname{Id}$  is the identity operator and we take  $P_{\text{hi}} := P_{\gtrsim N}$ . Thus, by commuting Fourier multiplier operators, using the self-adjointness of Littlewood-Paley operators and duality properties of Sobolev norms, we have

$$\begin{aligned} \left| \operatorname{Im} \int_{\mathbb{T}_\lambda} (If + f) \partial_x (\overline{If} - \overline{f}) dx \right| &\leq |\langle P_{\text{hi}}(If + f), (I - \operatorname{Id}) \partial_x \overline{f} \rangle_{L^2(\mathbb{T}_\lambda)}| \\ &\leq \|P_{\text{hi}}(If + f)\|_{H^{\frac{1}{2}}} \|P_{\text{hi}}(I - \operatorname{Id}) \partial_x f\|_{H^{-\frac{1}{2}}} \\ &\leq \left( \|P_{\text{hi}} If\|_{H^{\frac{1}{2}}} + \|P_{\text{hi}} f\|_{H^{\frac{1}{2}}} \right)^2. \end{aligned}$$



Since  $1 \lesssim N^{-\frac{1}{2}} \langle k \rangle^{\frac{1}{2}}$ ,  $1 \lesssim N^{-\frac{1}{2}} m(k) \langle k \rangle^{\frac{1}{2}}$  for all  $|k| \gtrsim N$ , we have

$$\|P_{\text{hi}} I f\|_{H^{\frac{1}{2}}} \lesssim N^{-\frac{1}{2}} \|I f\|_{H^1}, \quad (7.6)$$

$$\|P_{\text{hi}} f\|_{H^{\frac{1}{2}}} \lesssim N^{-\frac{1}{2}} \|I f\|_{H^1}. \quad (7.7)$$

Thus the first term in the right hand side of (7.5) is bounded by  $N^{-1} \|I f\|_{H^1}^2$ .

For the second term in the right hand side of (7.5), we write

$$|I f|^4 - |f|^4 = |I f|^2 I f (\overline{I f} - \overline{f}) + |I f|^2 (I f - f) \overline{f} + I f (\overline{I f} - \overline{f}) |f|^2 + (I f - f) \overline{f} |f|^2$$

and we treat, for example, the second term (modulo complex conjugation, it has all three possible factors involved); the others can be argued for analogously. By Hölder's inequality,

$$\begin{aligned} \left| \int_{\mathbb{T}_\lambda} |I f|^2 \overline{f} P_{\text{hi}}(I - \text{Id}) f \, dx \right| &\leq |\langle (I - \text{Id}) f, P_{\text{hi}}(|I f|^2 f) \rangle_{L^2}| \\ &\lesssim \|(I - \text{Id}) f\|_{L^6} \|P_{\text{hi}}(|I f|^2 f)\|_{L^{\frac{6}{5}}}. \end{aligned}$$

Then, by Sobolev embedding,

$$\|(I - \text{Id}) f\|_{L^6} \lesssim \|P_{\text{hi}}(I - \text{Id}) f\|_{H^{\frac{1}{3}}} \leq \|P_{\text{hi}} I f\|_{H^{\frac{1}{2}}} + \|P_{\text{hi}} f\|_{H^{\frac{1}{2}}} \quad (7.8)$$

and we can use the estimates (7.6)-(7.7) to gain a factor of  $N^{-\frac{1}{2}}$ . Another decaying factor is obtained via a Bernstein estimate, and then by Leibniz and Hölder inequalities, we get

$$\begin{aligned} \|P_{\text{hi}}(|I f|^2 f)\|_{L^{\frac{6}{5}}} &\lesssim N^{-\frac{1}{2}} \|J_x^{\frac{1}{2}} P_{\text{hi}}(|I f|^2 f)\|_{L^{\frac{6}{5}}} \\ &\lesssim N^{-\frac{1}{2}} \left( \|J_x^{\frac{1}{2}} I f\|_{L^2} \|I f\|_{L^6} \|f\|_{L^6} + \|J_x^{\frac{1}{2}} f\|_{L^2} \|I f\|_{L^6}^2 \right) \\ &\lesssim N^{-\frac{1}{2}} \|I f\|_{H^1}^3, \end{aligned} \quad (7.9)$$

where in the last step we used the Sobolev embedding as in (7.8) and  $\|I f\|_{H^{\frac{1}{2}}} \lesssim \|f\|_{H^{\frac{1}{2}}} \lesssim \|I f\|_{H^1}$ . Notice that if we do not drop the frequency restriction when passing to (7.9), at least one factor (in both terms) has to be supported on frequencies  $\gtrsim N$ , hence by arguing as for (7.6), we could get another factor of  $N^{-\frac{1}{2}}$ . Therefore, we obtain that the second term of (7.5) is bounded by  $N^{-\frac{3}{2}} \|I f\|_{H^1}^4$ .  $\square$

## 8. PROOF OF PROPOSITION 2.14 VIA THE $I$ -METHOD

In order to prove that blowup of the  $H^{\frac{1}{2}}$ -norm of a solution  $v$  of (2.43) does not occur in finite time, we adapt the  $I$ -method of [7, 8] (therein also referred to as “the almost conserved energy method”) to also incorporate the almost conservation of  $\mathcal{P}[I v]$ .

For initial data  $v_0 \in \mathcal{H}^s(\mathbb{T}) := \{f \in H^s(\mathbb{T}) : M[f] < 4\pi\}$ ,  $s < 1$ , its energy  $E[v_0]$  might not even be defined. However, the functionals  $\mathcal{E}[I v(t)]$  and  $\mathcal{P}[I v(t)]$  are well-defined and via Lemma 3.3,

$$\|I v(t)\|_{H^1}^2 \lesssim |\mathcal{E}[I v(t)]| + \mathcal{P}[I v(t)]^2 + 1, \quad (8.1)$$

where the smoothing operator  $I$  is defined by (2.50) in Section 2.4 and  $v$  is a (local) solution of (2.43) with  $v(0) = v_0$ . This control allows us to iterate the local well-posedness theory for any initial data in  $\mathcal{H}^s(\mathbb{T})$  and prove that the solution  $v$  exists for arbitrarily large times.

Since (2.54) allows for  $\|I v_0\|_{\dot{H}^1} \sim N^{1-s}$ , which would give a time of existence  $\delta \downarrow 0$  as  $N \uparrow \infty$ , we use the scaling transformation (1.5) and we note that

$$\|I v_0^\lambda\|_{\dot{H}^1(\mathbb{T}_\lambda)} \lesssim N^{1-s} \lambda^{-s} \|v_0\|_{\dot{H}^s(\mathbb{T})}. \quad (8.2)$$

We choose the scaling parameter

$$\lambda = N^{\frac{1-s}{s}} \quad (8.3)$$

to ensure that  $\delta \gtrsim 1$  uniformly in  $N$  and  $\lambda$ . We then have  $1 \ll \lambda \leq N$  in the regularity range  $\frac{1}{2} \leq s < 1$ , (in particular,  $\lambda = N$  for  $s = \frac{1}{2}$ ). We also record that  $\|v_0^\lambda\|_{H^s(\mathbb{T}_\lambda)}$ ,  $P[Iv_0^\lambda]$ ,  $E[Iv_0^\lambda]$  are bounded by constants depending only on  $\|v_0\|_{H^s(\mathbb{T})}$ .

A slightly modified iteration argument concludes the proof of Proposition 2.14. Indeed, consider  $B > 0$  such that

$$B^2 \sim \|v_0\|_{H^s(\mathbb{T})}^2 + |\mathcal{E}[Iv_0]| + \mathcal{P}[Iv_0]^2 + 1$$

and suppose that at step  $j$ , we have

$$\|Iv^\lambda(j\delta)\|_{H^1(\mathbb{T}_\lambda)} \leq B.$$

Then, by Proposition 4.1,

$$\|Iv^\lambda\|_{Z^1([j\delta, j\delta+\delta] \times \mathbb{T}_\lambda)} \leq D$$

and according to Proposition 6.11,

$$|\mathcal{E}^3[v^\lambda(j\delta + \delta)]| \leq |\mathcal{E}^3[v^\lambda(j\delta)]| + \delta^{0+} N^{-\gamma} \lambda^{-\kappa} C_1(D)$$

with  $\gamma = \frac{3}{2}-$ ,  $\kappa = 1-$ . Assuming that we run this iteration  $J$  times so that we cover the scaled time interval  $[0, \lambda^2 T]$ , i.e. assuming that we choose  $J$  such that

$$J \gtrsim \lambda^2 T, \quad (8.4)$$

we have

$$|\mathcal{E}^3[v^\lambda(J\delta)]| \leq |\mathcal{E}^3[v^\lambda(0)]| + J\delta^{0+} N^{-\gamma} \lambda^{-\kappa} C_1(D).$$

Notice that  $|\mathcal{E}^3[v(t)]|$  stays bounded (e.g.  $|\mathcal{E}^3[v(t)]| \leq 2|\mathcal{E}^3[v^\lambda(0)]|$ ) over the entire  $[0, \lambda^2 T]$  if we further impose that  $N$  is chosen such that

$$J \lesssim N^\gamma \lambda^\kappa. \quad (8.5)$$

At each iteration step, due to Proposition 7.3 and Proposition 7.4, we have in particular that

$$|\mathcal{E}[Iv^\lambda((j+1)\delta)]| \leq 2|\mathcal{E}^3[v^\lambda(0)]| + N^{-1+} C_2(D), \quad (8.6)$$

$$|\mathcal{P}[Iv^\lambda((j+1)\delta)]| \leq |\mathcal{P}[v^\lambda(0)]| + N^{-1} C_3(D), \quad (8.7)$$

where we used a version of (2.11) restricted to the time interval  $[j\delta, (j+1)\delta]$ . We get

$$\|Iv^\lambda((j+1)\delta)\|_{H^1(\mathbb{T}_\lambda)} \lesssim D.$$

We choose  $N$  large enough so that in (8.6) and (8.7) the second terms are dominated by the first terms. By Lemma 3.3, we then deduce

$$\|Iv^\lambda((j+1)\delta)\|_{H^1(\mathbb{T}_\lambda)} \leq B$$

and thus we get to perform the iteration again.

Note that (8.4), (8.5) and  $s \geq \frac{1}{2}$  yield

$$T \lesssim N^{\gamma-(\kappa-2)+\frac{1}{s}(\kappa-2)} \lesssim N^{\gamma+\kappa-2}.$$

In our case,  $\gamma + \kappa - 2 = \frac{1}{2}-$ ; hence, given any large  $T$ , we can choose a frequency threshold  $N = N(T) \gg 1$  for the  $I$ -operator.

Notice that for all  $t \in [0, \lambda^2 T] \subset [0, J\delta]$ , we have  $\mathcal{E}[Iv^\lambda(t)] \lesssim \mathcal{E}[Iv_0^\lambda] \lesssim 1$  and  $\mathcal{P}[Iv^\lambda(t)] \lesssim \mathcal{P}[Iv_0^\lambda] \lesssim 1$ , thus  $\|Iv^\lambda(t)\|_{H^1(\mathbb{T}_\lambda)} \lesssim 1$ . Also, we recall that we owe to undo the scaling:

$$\|v(t)\|_{H^s(\mathbb{T})} \lesssim \lambda^s \|v^\lambda(\lambda^2 t)\|_{H^s(\mathbb{T}_\lambda)} \lesssim \lambda^s \|Iv^\lambda(t)\|_{H^1(\mathbb{T}_\lambda)} \lesssim N^{1-s},$$

for all  $t \in [0, T]$ , where we used (2.53) and (8.3). The above numerology allows us to take  $N \sim T^{2+}$  and thus

$$\sup_{t \in [0, T]} \|v(t)\|_{H^s(\mathbb{T})} \lesssim T^{2-2s+}$$

for any  $\frac{1}{2} \leq s < 1$ .

#### APPENDIX A. MILD ILL-POSEDNESS BELOW $H^{\frac{1}{2}}(\mathbb{T})$

The scope of this section is to provide the analogue to the periodic setting of the ill-posedness result of Biagioni and Linares [2] where it was shown that the flow map  $u_0 \mapsto u$  of (1.1) is not uniformly continuous from bounded subsets of  $H^s(\mathbb{R})$  into  $C_t H_x^s([-T, T] \times \mathbb{R})$ , for any  $T > 0$  and  $0 \leq s < \frac{1}{2}$ . However, the method (which was introduced in [21]) uses the family of soliton solutions of (1.1) on  $\mathbb{R}$  (see [19, 46]), for which the corresponding initial data are not compactly supported, hence this strategy cannot be adapted to the periodic setting.

We also recall an observation of Grünrock and Herr [12, Remark 2], that in the periodic setting, due to the presence of a translation in space operator in the gauge transformation  $\mathcal{G}^\beta$  (see (2.42)), at any regularity level, the uniform continuity of the solution map of (1.1) fails without fixing the mass on bounded subsets of  $H^s(\mathbb{T})$  (see also [18, Theorem 3.1.1.(ii)]). Nevertheless, for the gauge equivalent equation (2.43) one does not face the uniform continuity bottleneck due to the translation operator and it was for this equation that the contraction mapping argument was applied in [17].

Using ideas similar to those in [4, 6], we construct smooth solutions that prove the failure of uniform continuity of the solution map of (2.43) on bounded subsets of  $H^s(\mathbb{T})$ , for  $0 \leq s < \frac{1}{2}$ . Since the solutions we construct are supported on single frequencies (monochromatic waves), the same result holds true for the Fourier-Lebesgue spaces  $\mathcal{FL}^{s,r}(\mathbb{T})$  with  $s < \frac{1}{2}$ ,  $r \geq 1$ .

**Lemma A.1.** *Let  $0 \leq s < \frac{1}{2}$  and  $T > 0$ . For any  $0 < \delta \ll \varepsilon < 1$ , there exist smooth initial data  $v_0, \tilde{v}_0$  such that*

$$\|v_0\|_{H^s(\mathbb{T})}, \|\tilde{v}_0\|_{H^s(\mathbb{T})} \lesssim \varepsilon, \tag{A.1}$$

$$\|v_0 - \tilde{v}_0\|_{H^s(\mathbb{T})} \lesssim \delta, \tag{A.2}$$

and for which the corresponding solutions  $v, \tilde{v}$  to (2.43) have the property

$$\|v - \tilde{v}\|_{L_t^\infty([-T, T]; H_x^s(\mathbb{T}))} \gtrsim \varepsilon. \tag{A.3}$$

*Proof.* Let  $a \in \mathbb{C}$  and  $N \in \mathbb{Z}$ ,  $N \gg 1$  (to be chosen later) and consider functions supported on a single frequency of the form

$$v_{N,a}(t, x) = ae^{i(Nx + \theta(N)t)},$$

for some  $\mathbb{R}$ -valued  $\theta(\cdot)$ . We have

$$\mu[v_{N,a}] = |a|^2, \quad \psi[v_{N,a}] = -2|a|^2 N + \frac{1}{2}|a|^4$$

and thus we compute the corresponding nonlinearity of (2.43):

$$\mathcal{N}(v_{N,a}) = |a|^2 a N e^{i(Nx + \theta(N)t)}.$$

Then, by taking  $\theta(N) = -N^2 - |a|^2 N$ , the function

$$v_{N,a}(t, x) = a e^{i(Nx - N^2 t - |a|^2 N t)} \quad (\text{A.4})$$

is a solution of (2.43) with

$$\|v_{N,a}(t, x)\|_{L_x^2(\mathbb{T})} \sim |a|, \quad \|v_{N,a}(t, x)\|_{\dot{H}_x^s(\mathbb{T})} \sim |a| N^s$$

and since  $s \geq 0$ , we also have

$$\|v_{N,a}(t, x)\|_{H_x^s(\mathbb{T})} \sim |a| N^s.$$

Now let  $a = bN^{-s}$  and  $\tilde{a} = \tilde{b}N^{-s}$  with  $b, \tilde{b} \in \mathbb{C}$  such that  $|b| \sim |\tilde{b}| \sim \varepsilon$  and  $|b - \tilde{b}| \lesssim \delta$ . We find

$$\|v_{N,a}(0, x) - v_{N,\tilde{a}}(0, x)\|_{H_x^s(\mathbb{T})} = |b - \tilde{b}| N^{-s} \|e^{iNx}\|_{H_x^s(\mathbb{T})} \lesssim \delta.$$

On the other hand, by setting  $\varphi(N, b) := |bN^{-s}|^2 N$  to simplify the writing, we obtain

$$\begin{aligned} \|v_{N,a}(t, x) - v_{N,\tilde{a}}(t, x)\|_{H_x^s(\mathbb{T})} &= \left| b e^{-i\varphi(N,b)t} - \tilde{b} e^{-i\varphi(N,\tilde{b})t} \right| N^{-s} \|e^{iNx}\|_{H_x^s(\mathbb{T})} \\ &\gtrsim |b| \left| e^{-i\varphi(N,b)t} - e^{-i\varphi(N,\tilde{b})t} \right| - |b - \tilde{b}| \\ &\gtrsim \varepsilon \left| e^{i(\varphi(N,b) - \varphi(N,\tilde{b}))t} - 1 \right| - \delta. \end{aligned}$$

Note that

$$\varphi(N, b) - \varphi(N, \tilde{b}) = N^{1-2s}(|b|^2 - |\tilde{b}|^2) \quad (\text{A.5})$$

and that at  $t = t_N$ , where

$$t_N := \frac{\pi}{\varphi(N, b) - \varphi(N, \tilde{b})}, \quad (\text{A.6})$$

the two solutions have opposite phases, and thus

$$\|v_{N,a}(t_N, x) - v_{N,\tilde{a}}(t_N, x)\|_{H_x^s(\mathbb{T})} \gtrsim \varepsilon - \delta \sim \varepsilon.$$

Indeed, since the power of  $N$  in (A.5) is positive, we can choose an integer  $N = N(\varepsilon, T)$  (independent of  $\delta$ ) such that  $|t_N| \leq T/2$ , or equivalently

$$|\varphi(N, b) - \varphi(N, \tilde{b})| \gtrsim T^{-1}.$$

□

**Remark A.2.** One can easily adapt the above argument to any other gauge equivalent equation, including (1.1) itself. Indeed, it is enough to take

$$\theta_\beta(N) = \theta(N) - (\beta^2 - \frac{3}{2}\beta + \frac{1}{2})|a|^4.$$

Correspondingly, we take

$$\varphi_\beta(N, b) = \varphi(N, b) - (\beta^2 - \frac{3}{2}\beta + \frac{1}{2})|b|^4 N^{-4s}$$

and note that for  $N \gg 1$ , the difference in phase is essentially as above, i.e.

$$\varphi_\beta(N, b) - \varphi_\beta(N, \tilde{b}) \sim \varphi(N, b) - \varphi(N, \tilde{b}).$$

## REFERENCES

- [1] M. AGUEH, *Sharp Gagliardo–Nirenberg inequalities and mass transport theory*, J. Dyn. Differ. Equ., 18 (2006), pp. 1069–1093.
- [2] H. BIAGIONI AND F. LINARES, *Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations*, Trans. Amer. Math. Soc., 353 (2001), pp. 3649–3659.
- [3] J. BOURGAIN, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations*, Geom. Func. Anal., (1993).
- [4] N. BURQ, P. GÉRARD, AND N. TZVETKOV, *An instability property of the nonlinear Schrödinger equation on  $S^d$* , Math. Res. Lett., 9 (2002), pp. 323–336.
- [5] Y. CHER, G. SIMPSON, AND C. SULEM, *Local structure of singular profiles for a derivative nonlinear Schrödinger equation*, SIAM J. Appl. Dyn. Syst., 16 (2016), pp. 514–545.
- [6] M. CHRIST, J. COLLIANDER, AND T. TAO, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math., (2003), pp. 1235–1293.
- [7] J. COLLIANDER, M. KEEL, G. STAFFILANI, H. TAKAOKA, AND T. TAO, *Global well-posedness for Schrödinger equations with derivative*, SIAM J. Math. Anal., 33 (2001), pp. 649–669.
- [8] ———, *A refined global well-posedness for Schrödinger equations with derivative*, SIAM J. Math. Anal., 34 (2002), pp. 64–86.
- [9] ———, *Multilinear estimates for periodic KdV equations, and applications*, J. Funct. An., 211 (2004), pp. 173–218.
- [10] D. DE SILVA, N. PAVLOVIĆ, G. STAFFILANI, AND N. TZIRAKIS, *Global well-posedness for a periodic nonlinear Schrödinger equation in 1D and 2D*, Discrete Contin. Dyn. Syst., 19 (2007), pp. 37–65.
- [11] A. GRÜNROCK, *Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS*, Int. Math. Res. Not., 2005 (2005), pp. 2525–2558.
- [12] A. GRÜNROCK AND S. HERR, *Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data*, SIAM J. Math. Anal., 39 (2008), pp. 1890–1920.
- [13] Z. GUO AND Y. WU, *Global well-posedness for the derivative nonlinear Schrödinger equation in  $H^{\frac{1}{2}}(\mathbb{R})$* , Discrete Contin. Dyn. Syst., 37 (2017), pp. 257–264.
- [14] N. HAYASHI, *The initial value problem for the derivative nonlinear Schrödinger equation in the energy space*, Nonlinear Anal., 20 (1993), pp. 823–833.
- [15] N. HAYASHI AND T. OZAWA, *On the derivative nonlinear Schrödinger equation*, Physica D., 55 (1992), pp. 14–36.
- [16] ———, *Finite energy solutions of nonlinear Schrödinger equations of derivative type*, SIAM J. Math. Anal., 25 (1994).
- [17] S. HERR, *On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition*, Int. Math. Res. Not., (2006).
- [18] ———, *Well-posedness results for dispersive equations with derivative nonlinearities*, PhD thesis, University of Dortmund, 2006.
- [19] D. KAUP AND A. NEWELL, *An exact solution for a derivative nonlinear Schrödinger equation*, J. Math. Phys., 19 (1978), pp. 789–801.
- [20] C. KENIG, G. PONCE, AND L. VEGA, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc., 9 (1996), pp. 573–603.
- [21] C. E. KENIG, G. PONCE, AND L. VEGA, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J., 106 (2001), pp. 617–633.
- [22] S. KWON AND Y. WU, *Orbital stability of solitary waves for derivative nonlinear Schrödinger equation*, J. Anal. Math., (in press).
- [23] S. LE COZ AND Y. WU, *Stability of multisolitons for the derivative nonlinear Schrödinger equation*, Int. Math. Res. Not., (2017).
- [24] J. LEBOWITZ, H. ROSE, AND E. SPEER, *Statistical mechanics of the nonlinear Schrödinger equation*, J. Stat. Phys., 50 (1988), pp. 657–687.

- [25] J. LEE, *Analytic properties of Zakharov-Shabat inverse scattering problem with a polynomial spectral dependence of degree 1 in the potential*, PhD thesis, Yale University, 1983.
- [26] J. LEE, *Global solvability of the derivative nonlinear Schrödinger equation*, Trans. Amer. Math. Soc., 314 (1989), pp. 107–118.
- [27] J. LIU, P. PERRY, AND C. SULEM, *Global existence for the derivative nonlinear Schrödinger equation by the method of inverse scattering*, Comm. Partial Differential Equations, 41 (2016), pp. 1692–1760.
- [28] X. LIU, G. SIMPSON, AND C. SULEM, *Focusing singularity in a derivative nonlinear Schrödinger equation*, Physica D: Nonlinear Phenomena, 262 (2013), pp. 48–58.
- [29] C. MIAO, X. TANG, AND G. XU, *Solitary waves for nonlinear Schrödinger equation with derivative*, Commun. Contemp. Math., (2017).
- [30] ———, *Stability of the traveling waves for the derivative Schrödinger equation in the energy space*, Calc. Var., 56:45 (2017).
- [31] C. MIAO, Y. WU, AND G. XU, *Global well-posedness for Schrödinger equation with derivative in  $H^{1/2}(\mathbb{R})$* , J. Differential Equations, 251 (2011), pp. 2164–2195.
- [32] K. MIO, T. OGINO, K. MINAMI, AND S. TAKEDA, *Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas*, J. Phys. Soc. Japan, 41 (1976), pp. 265–271.
- [33] E. MJOLHUS, *On the modulational instability of hydromagnetic waves parallel to the magnetic field*, J. Plasma Physics, 16 (1976), pp. 321–334.
- [34] R. MOSINCAT AND T. OH, *A remark on global well-posedness of the derivative nonlinear Schrödinger equation on the circle*, C.R. Acad. Sci. Paris, Ser. I, 353 (2015), pp. 837–841.
- [35] A. R. NAHMOD, T. OH, L. REY-BELLET, AND G. STAFFILANI, *Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS*, J. Eur. Math. Soc., (2010).
- [36] D. PELINOVSKY AND Y. SHIMABUKURO, *Existence of global solutions to the derivative NLS equation with the inverse scattering transform method*, Int. Math. Res. Not., (in press).
- [37] A. REGISTER, *Parallel propagation of nonlinear low-frequency waves in high- $\beta$  plasma*, Phys. Fluids, 14 (1971).
- [38] H. TAKAOKA, *Well-posedness for the one dimensional Schrödinger equation with the derivative nonlinearity*, Adv. Diff. Eqs., 4 (1999), pp. 561–680.
- [39] ———, *Global well-posedness for Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces*, Electron. J. Differential Equations, 42 (2001).
- [40] ———, *Energy transfer model for the derivative nonlinear Schrödinger equations on the torus*, arXiv:1603.01947, (2016).
- [41] ———, *A priori estimates and weak solutions for the derivative nonlinear Schrödinger equation on torus below  $H^{\frac{1}{2}}$* , J. Differential Equations, 260 (2016), pp. 818–859.
- [42] S. B. TAN, *Blow-up solutions for mixed nonlinear Schrödinger equations*, Acta Math. Sin., 20 (2004), pp. 115–124.
- [43] T. TAO, *Nonlinear Dispersive Equations: Local and Global Analysis*, no. 106 in CBMS, Amer. Math. Soc., 2007.
- [44] M. TSUTSUMI AND I. FUKUDA, *On solutions of the derivative nonlinear Schrödinger equation: Existence and uniqueness theorems*, Funkcial. Ekvac., 23 (1980), pp. 259–277.
- [45] ———, *On solutions of the derivative nonlinear Schrödinger equation II*, Funkcial. Ekvac., 24 (1981), pp. 85–94.
- [46] W. VAN SAARLOOS AND P. HOHENBERG, *Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equations*, Physica D: Nonlinear Phenomena, 56 (1992), pp. 303–367.
- [47] M. I. WEINSTEIN, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys., 87 (1983), pp. 567–576.
- [48] Y. WIN, *Unconditional uniqueness of the derivative nonlinear Schrödinger equation in energy space*, J. Math. Kyoto Univ., 48 (2008), pp. 683–697.
- [49] Y. WIN, *Global well-posedness of the derivative nonlinear Schrödinger equations on  $T$* , Funkcial. Ekvac., 53 (2010), pp. 51–88.

- [50] Y. WU, *Global well-posedness of the derivative nonlinear Schrödinger equations in energy space*, Anal. PDE, 6 (2013), pp. 1989–2002.
- [51] Y. WU, *Global well-posedness on the derivative nonlinear Schrödinger equation*, Anal. PDE, 8 (2015), pp. 1101–1112.